THE ESSENTIAL SPECTRUM OF ADVECTIVE EQUATIONS

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Abstract. The geometric optics stability method is extended to a general class of linear advective PDE's with pseudodifferential bounded perturbation. We give a new short proof of Vishik's formula for the essential spectral radius. We show that every point in the dynamical spectrum of the corresponding bicharacteristic-amplitude system contributes a point into the essential spectrum of the PDE. Generic spectral pictures are obtained in Sobolev spaces of sufficiently large smoothness. Applications to instability are presented.

Contents

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1. Introduction

The subject of this article is rooted in the geometric stability method for ideal fluids developed in the early 90's by Friedlander and Vishik [14, 46], and independently by Lifschitz and Hameri [26, 27]. Studying localized shortwave instabilities of a general steady flow u_0 one is naturally led to consider solutions of the linearized Euler equation in the WKB form

$$f(x,t) = b(x,t)e^{iS(x,t)/\delta} + O(\delta) \tag{1}$$

where δ is a small parameter. For a very limited class of flows u_0, such a solution may become an exact solution to the (even nonlinear!) equation for a finite δ. Well-known classical examples were provided by Craik and Criminale in [8], in the case of a linear vector field u_0. For the flow with elliptic streamlines genuinely three-dimensional perturbations of the form (1) are found to be unstable as shown by numerical calculations of Pierrehumbert [34] and Bayly [3]. This elliptic instability is believed to be an integral part of transition to turbulence in certain laminar flows [4, 19, 31].

For general equilibrium no explicit solution is available. We substitute (1) into the linearized Euler equation and set equal the leading order terms on both sides. This gives us evolution laws for the frequency $\xi = \nabla S$ and amplitude b. Written in Lagrangian coordinates associated with the basic flow u_0, they form a system of ODE's, called the bicharacteristic-amplitude system, given by

$$x_t = u(x), \tag{2a}$$

$$\xi_t = -\partial u^\top(x)\xi, \tag{2b}$$

$$b_t = -\partial u(x)b + 2\frac{\xi\otimes\xi}{|\xi|^2}\partial u(x)b. \tag{2c}$$

The main result of [14, 46, 26, 27] states that the exponential growth type of the semigroup generated by the Euler equation dominates the maximal Lyapunov exponent of the amplitude equation (2c). This

provides a sufficient condition for exponential instability of the general steady flow u_0. Papers [15, 24] exhibit a number of examples for which this condition applies successfully, mainly due to its local nature. In particular, it is shown that any flow with exponential stretching, such as a flow with hyperbolic stagnation point, is unstable.

Unlike classical normal modes in a periodic domain, shortwave perturbations (1) are linked to the essential (continuous) spectrum rather than the point spectrum. It was proved by Vishik [45] that for the essential spectral radius $r_{\mathrm{ess}}(\mathbf{G}_t)$ of the semigroup operator $\mathbf{G}_t$ the following formula holds:

$$r_{\mathrm{ess}}(\mathbf{G}_t) = e^{t\mu}, \tag{3}$$

where μ is the maximal Lyapunov exponent of the amplitude b. Subsequently, Shvydkoy and Vishik [42] have shown that, in fact, for any given Lyapunov exponent λ of the amplitude equation, the circle of radius $e^{t\lambda}$ contains a point of the essential spectrum.

The geometric optics method has been applied to many other non-dissipative equations of ideal hydrodynamics, such as Boussinesq approximation [14], SQG [13], Euler in vorticity form [23, 24]. Equations with Coriolis forcing were treated in [16, 43].

The purpose of this present paper is twofold. First, we introduce a general class of equations, which includes all the equations mentioned above. In these settings we derive the bicharacteristic-amplitude system and give a new short proof of Vishik's formula (3). Second, we give a detailed description of the essential spectrum in Sobolev spaces.

We consider the following first order linear PDE, which we call an advective PDE:

$$f_t = -(u \cdot \nabla) f + \mathbf{A} f, \tag{4}$$

where u is a time-independent smooth vector field and $\mathbf{A}$ is a pseudo-differential operator of zero order. We consider 2π-periodic boundary conditions. For instance, the Euler equation for incompressible ideal fluid linearized about a steady state u can be represented in form (4), where $\mathbf{A}$ has principal symbol

$$\mathbf{a}_0(x,\xi) = -\partial u(x) + 2\frac{\xi \otimes \xi}{|\xi|^2}\partial u(x). \tag{5}$$

We recognize in (5) the right hand side of the amplitude equation (2c). In Section 3 we show that for any advective equation the amplitude of a shortwave perturbation evolves according to the following ODE

$$b_t = \mathbf{a}_0(x(t), \xi(t)) b, \tag{6}$$

where $\mathbf{a}_0$ is the principal symbol of $\mathbf{A}$, and $(x(t), \xi(t))$ is the phase flow of (2a), (2b). We regard (6) as a dynamical system over this phase flow. Due to the Oseledets Multiplicative Ergodic Theorem, we can consider the set of Lyapunov exponents, of which the maximal one determines the essential spectral radius over of the semigroup generated by (4) over L^2 via formula (3). This is proved in Theorem 4.1. In fact, we prove a version of (3) for any energy-Sobolev space $H^m = W^{2,m}$, $m \in \mathbb{R}$. In this case the amplitude cocycle is to be augmented by the frequency, i.e. we consider a new cocycle $|\xi(t)|^m b(t)$, which we call the $b\xi^m$-cocycle.

Sections 5 and 6 are devoted to more detailed description of the essential spectrum. We show that the dynamical spectrum (also called Sacker-Sell spectrum) of the $b\xi^m$-cocycle, in a sense, forms the skeleton of the essential spectrum. In Theorem 5.3 we prove the following inclusions:

$$\exp\{t\Sigma_m\} \subset |\sigma_{\mathrm{ess}}(\mathbf{G}_t)| \subset \exp\{t[\min \Sigma_m, \max \Sigma_m]\}, \tag{7}$$

where Σ_m is the dynamical spectrum of the $b\xi^m$-cocycle. According to a theorem of Sacker and Sell [35], the dynamical spectrum of a d-dimensional system is the union of at most d segments on the real line. So, when dimension of the system is one, the dynamical spectrum is connected. In this case, inclusions (7) turn into exact identities. This situation applies, for instance, to all gradient systems of the form (4), or to the 2D Euler equation in both vorticity and velocity form.

The case of large smoothness parameter m is treated in Section 6. In this case a much more refined description of the spectrum will be given. If the basic flow u has exponential stretching of trajectories, the $|\xi|^m$-component of the $b\xi^m$-cocycle becomes more influential, and eventually takes control over the spectrum of the whole cocycle. This leads to two favorable consequences. First, we can control the asymptotics of the end-points of Σ_m with $|m| \to \infty$, and second, starting from a certain point Σ_m becomes connected due to the fact that the cocycle $|\xi|^m$ itself is one-dimensional. The precise quantitative condition on $|m|$ is stated in terms of relevant Lyapunov exponents in Theorem 6.1. In summary, we will prove the following result.

Theorem 1.1. *Suppose u has exponential stretching of trajectories, and let $|m|$ be large enough. Let us denote $s = \sup_{k\in\mathbb{R}}\{\min \Sigma_k\}$ and $S = \inf_{k\in\mathbb{R}}\{\max \Sigma_k\}$. Then the following holds:*

1) Σ_m *is connected;*
2) $\min \Sigma_m < s$ *and* $S < \max \Sigma_m$ *;*
3) $|\sigma_{\mathrm{ess}}(\mathbf{G}_t)| = \exp\{t\Sigma_m\}$*;*
4) $\mathbb{T} \cdot \exp\{t[\min \Sigma_m, s] \cup [S, \max \Sigma_m]\} \subset \sigma_{\mathrm{ess}}(\mathbf{G}_t)$ *;*

Thus, as we see, $\sigma_{\mathrm{ess}}(\mathbf{G}_t)$ has no circular gaps, and contains solid outer and inner rings (see Figure 1 for generic spectral picture).

Parallel results will proved for the spectrum of the generator (the RHS) of the advective equation (4). In this case we restrict the cocycles to the invariant subset $\xi \cdot u(x) = 0$. We show that under the assumptions of Theorem 1.1 the dynamical spectrum of the restricted $b\xi^m$-cocycle coincides with the original spectrum Σ_m. Thus, in addition to the above the following properties will be proved for the generator $\mathbf{L}$ (see Theorem 6.2):

5) $[\min \Sigma_m, s] \cup [S, \max \Sigma_m] + i\mathbb{R} \subset \sigma_{\mathrm{ess}}(\mathbf{L})$;
6) $\mathrm{Re}\, \sigma_{\mathrm{ess}}(\mathbf{L}) = [\min \Sigma_m, \max \Sigma_m]$.

In particular, this implies the Annular Hull Theorem 6.3, and other desirable spectral properties.

In the case of the energy space L^2, such results are not yet available. The idea of considering the restricted cocycle has been already exploited by Latushkin and Vishik [20] in an attempt to prove the identity between spectral bounds of the semigroup and generator of the 3D Euler equation. In the 2D case, however, this result is proved, and as matter of fact, a complete description of the spectra is given in [41, 40]. These are the solid annulus and vertical strip, respectively, for any $m \neq 0$. The same spectral picture has been found for the SQG equation in [13]. What these equations have in common is that their b-cocycles have trivial dynamical spectrum $\Sigma_0 = \{0\}$. In Section 6 we show that any advective equation with trivial dynamical spectrum has annulus–strip essential spectrum.

Section 7 contains the proof of the above results. Our main tool is the theory of linear cocycles and Mañe sequences. Some of our statements are novel and have certain applications to the spectral theory of Mather semigroups. We present these results in a separate paper [38] that will be published elsewhere.

2. Formulation

Let $u(x)$ be a smooth vector field on the n-dimensional torus $\mathbb{T}^n$. Incompressibility of u will be our standing hypothesis, although it is not always necessary.

We study linear partial differential equations of the form

$$f_t = -(u \cdot \nabla) f + \mathbf{A} f, \quad t \geq 0 \tag{8}$$

subject to periodic boundary conditions

$$f(x + 2\pi e_i, t) = f(x, t), \quad i = 1, \dots, n,$$

where $\{e_i\}_{i=1}^n$ are the vectors of the standard unit basis. A solution $f(x,t)$ assumes values in $\mathbb{C}^d$, and $\mathbf{A}$ is a discrete pseudodifferential operator (PDO) defined on smooth functions by

$$\mathbf{A}f(x) = \sum_{k\in\mathbb{Z}^n\backslash\{0\}} \mathbf{a}(x,k)\hat{f}(k)e^{ik\cdot x}, \tag{9}$$

where $\mathbf{a}$ is a $d\times d$ matrix-valued symbol

$$\mathbf{a}(x,\xi) : \mathbb{C}^d \to \mathbb{C}^d, \tag{10}$$

defined for all $x\in\mathbb{T}^n$ and $\xi\in\mathbb{R}^n\backslash\{0\}$.

Throughout the text we impose the following smoothness and growth assumptions on $\mathbf{a}(x,\xi)$. The class $\mathcal{S}^m$, $m\in\mathbb{R}$, consists of all infinitely smooth symbols $\mathbf{a}(x,\xi)$, for $x\in\mathbb{T}^n$ and $\xi\in\mathbb{R}^n\backslash\{0\}$, such that for any multi-indices α and β there exists a constant $C_{\alpha,\beta}$ for which the following estimate holds

$$|\partial_\xi^\alpha\partial_x^\beta\mathbf{a}(x,\xi)| \le C_{\alpha,\beta}|\xi|^{m-|\alpha|}, \quad x\in\mathbb{T}^n, |\xi|\ge 1.$$

Unlike in the classical definition of Hörmander classes [17, 37] we prefer to allow singularity at $\xi=0$. As we will see, this is a typical feature of many examples arising in fluid mechanics. On $\mathbb{T}^n$ (as well as any other compact manifolds) the singularity can be removed by replacing the original symbol with a smooth cut-off $\mathbf{a}(x,\xi)(1-\gamma(\xi))$. This replacement does not effect the operator $\mathbf{A}$ so long as $\gamma(\xi)$ is supported inside the unit ball. We remark that PDO's of the form (9) with symbols smooth in ξ obey the same classical principles as in the $\mathbb{R}^n$ case (see [9]).

The class of all pseudodifferential operators of the form (9) with $\mathbf{a}\in\mathcal{S}^m$ will be denoted by $\mathcal{L}^m$.

Let $H^m(\mathbb{T}^n)$, $m\in\mathbb{R}$, denote the Sobolev space of $\mathbb{C}^d$-valued functions on the torus, defined as

$$H^m(\mathbb{T}^n) = \left\{ f : \|f\|^2_{H^m(\mathbb{T}^n)} = |\hat{f}(0)|^2 + \sum_{k\in\mathbb{Z}^n} |k|^{2m}|\hat{f}(k)|^2 < \infty \right\}. \tag{11}$$

By the standard boundedness principle for PDO's [37, Theorem 7.1] we have

$$\mathbf{A} : H^m(\mathbb{T}^n) \to H^{m-s}(\mathbb{T}^n) \tag{12}$$

for any $\mathbf{A}\in\mathcal{L}^s$, and any $s,m\in\mathbb{R}$. In fact, in the case of torus (12) holds without any smoothness assumption in the ξ-variable, which can be proved by an application of Minkowski's inequality.

A symbol $\mathbf{a}\in\mathcal{S}^0$ is called **semiclassical** if

$$\mathbf{a} = \mathbf{a}_0 + \mathbf{a}_1, \tag{13}$$

where $\mathbf{a}_0 \in \mathcal{S}^0$ is homogenous of degree 0 in ξ (i.e. $\mathbf{a}_0(x, t\xi) = \mathbf{a}_0(x, \xi)$), and $\mathbf{a}_1 \in \mathcal{S}^{-1}$. We call $\mathbf{a}_0$ the **principal symbol** of the operator $\mathbf{A}$. Thus, if $\mathbf{a}$ is semiclassical, then

$$\mathbf{A} = \mathbf{A}_0 + \mathbf{A}_1, \tag{14}$$

where $\mathbf{A}_i$ is the PDO with the symbol $\mathbf{a}_i$. Since $\mathbf{a}_1 \in \mathcal{S}^{-1}$, we see from (12) that $\mathbf{A}_1$ maps $H^m(\mathbb{T}^n)$ into $H^{m+1}(\mathbb{T}^n)$, which embeds back into $H^m(\mathbb{T}^n)$ compactly. Hence, $\mathbf{A}_1$ is a compact operator on $H^m(\mathbb{T}^n)$.

In this work we consider only semiclassical symbols of class $\mathcal{S}^0$ so that $\mathbf{A}$ defines a bounded operator on any Sobolev space, and decomposition (14) holds.

Let us denote the right hand side of (8) by

$$\mathbf{L}f = -(u \cdot \nabla)f + \mathbf{A}f. \tag{15}$$

It consists of the advective derivative $-(u \cdot \nabla)f$ and bounded perturbation $\mathbf{A}f$. The advective derivative generates a C_0-semigroup acting by the rule

$$f \to f \circ \varphi_{-t},$$

where $\varphi = \{\varphi_t(x)\}_{t\in\mathbb{R},\, x\in\mathbb{T}^n}$ is the integral flow of the field $u(x)$. Hence, $\mathbf{L}$ itself generates a C_0-semigroup (see Engel and Nagel [10]). Let us denote it by $\mathbf{G} = \{\mathbf{G}_t\}_{t\geq 0}$. In view of time reversibility of equation (8) the semigroup $\mathbf{G}$ is invertible, and hence, it is a group.

2.1. **Constraints.** We now introduce a special class of constraints.

We consider an arbitrary smooth linear bundle $\mathcal{F}$ over the space of non-zero frequencies $\mathbb{R}^n\backslash\{0\}$. Let us denote its fibers by $F(\xi) \subset \mathbb{C}^d$, and assume that $F(\xi)$ is 0-homogenous and infinitely smooth in the region $\xi \neq 0$. We separately consider a fiber at zero, $F(0) \subset \mathbb{C}^d$. We call $\mathcal{F}$ **frequency bundle**.

Given a frequency bundle $\mathcal{F}$, we say that a function f on $\mathbb{T}^n$ satisfies the **frequency constraints** determined by $\mathcal{F}$ if $\hat{f}(k) \in F(k)$, for all $k \in \mathbb{Z}^n$.

Let $\mathbf{p}(\xi) : \mathbb{C}^d \to \mathbb{C}^d$ denote the orthogonal projection onto $F(\xi)$. According to our assumptions on the fibers $F(\xi)$, $\mathbf{p}$ is a classical symbol of class $\mathcal{S}^0$. For example, the incompressibility constraint, $\operatorname{div} f = 0$, corresponds to

$$F(\xi) = \{b : b \cdot \xi = 0\}, \tag{16}$$

$$\mathbf{p}(\xi) = \mathbf{id} - \frac{\xi \otimes \xi}{|\xi|^2}. \tag{17}$$

We introduce the corresponding Sobolev spaces subject to constraints,

$$H^m_{\mathcal{F}}(\mathbb{T}^n) = \{f \in H^m(\mathbb{T}^n) : \hat{f}(k) \in F(k)\}, \tag{18}$$

and the orthogonal projection

$$\mathbf{\Pi} : H^m(\mathbb{T}^n) \to H^m_{\mathcal{F}}(\mathbb{T}^n) \tag{19}$$

$$\widehat{\mathbf{\Pi} f}(k) = \mathbf{p}(k)\hat{f}(k). \tag{20}$$

We use special notation $H^m_0(\mathbb{T}^n)$ for the space of mean-zero functions and $H^m_{\mathrm{div}}(\mathbb{T}^n)$ for divergence-free fields. If $m = 0$, we write $L^2_{\mathcal{F}}(\mathbb{T}^n)$, $L^2_0(\mathbb{T}^n)$, and $L^2_{\mathrm{div}}(\mathbb{T}^n)$.

In the sequel, if constraints are given, we assume that they are respected by equation (8). In other words, $\mathbf{G}$ leaves $H^m_{\mathcal{F}}(\mathbb{T}^n)$ invariant. We note that under this assumption we can still consider the semigroup $\mathbf{G}$ on the whole space $H^m(\mathbb{T}^n)$, which corresponds to solving (8) without any constraints.

2.2. **Essential spectrum.** We now briefly state the definition of essential spectrum used in this paper.

For any closed operator $\mathbf{T}$ on a Banach space X we use the following classification of the spectrum (following Browder [5]). A point $z \in \sigma(\mathbf{T})$ is called a point of the **discrete spectrum** if it satisfies the following conditions:

(DS1) z is an isolated point in $\sigma(\mathbf{T})$;
(DS2) z has finite multiplicity, i.e. $\bigcup_{r=1}^{\infty} \mathrm{Ker}(z - \mathbf{T})^r = N$ is finite dimensional in X;
(DS3) The range of $z - \mathbf{T}$ is closed.

Otherwise, z is called a point of the **essential spectrum**. Thus,

$$\sigma(\mathbf{T}) = \sigma_{\mathrm{ess}}(\mathbf{T}) \cup \sigma_{\mathrm{disc}}(\mathbf{T}). \tag{21}$$

We note that if $\mathbf{T}$ is bounded, then condition (DS3) follows from (DS1,DS2).

Let $r_{\mathrm{ess}}(\mathbf{T})$ denote the radius of $\sigma_{\mathrm{ess}}(\mathbf{T})$, and let $\mathcal{C}$ be the Calkin algebra over X. According to Nussbaum [30], we have

$$r_{\mathrm{ess}}(\mathbf{T}) = \lim_{n\to\infty} \|\mathbf{T}^n\|_{\mathcal{C}}^{1/n}. \tag{22}$$

Concerning spectrum of a semigroup we recall that the discrete part obeys the spectral mapping property:

$$\sigma_{\mathrm{disc}}(\mathbf{G}_t)\backslash\{0\} = e^{t\sigma_{\mathrm{disc}}(\mathbf{L})},\ t \geq 0, \tag{23}$$

while the essential part may fail to satisfy it. Generally, we only have the inclusion (see [10])

$$e^{t\sigma_{\mathrm{ess}}(\mathbf{L})} \subset \sigma_{\mathrm{ess}}(\mathbf{G}_t)\backslash\{0\}. \tag{24}$$

3. The bicharacteristic-amplitude system

3.1. **Derivation.** We now would like to investigate asymptotic behavior of solutions to (8) with initial data given by a highly oscillating wavelet localized near some point $x_0 \in \mathbb{T}^n$:

$$f_0(x) = b_0(x) e^{i\xi_0 \cdot x/\delta}. \tag{25}$$

We consider solution in the geometric optics form

$$f(x,t) = b(x,t) e^{iS(x,t)/\delta} + O(\delta), \tag{26}$$

where $\nabla_x S(x,t) \neq 0$, for all $x \in \mathbb{T}^n$ and $t \geq 0$. On the next step we extract evolution laws for the amplitude b and the phase (eikonal) S by substituting $f(x,t)$ into equation (8); but first, we need to find an asymptotic formula for $\mathbf{A}f$.

Theorem 3.1. *Suppose $\mathbf{A} \in \mathcal{L}^0$ is a pseudodifferential operator with semiclassical symbol $\mathbf{a}(x,\xi)$ so that decomposition* (13) *holds. Let*

$$f_\delta(x) = b(x) e^{iS(x)/\delta},$$

where b, $S \in C^\infty(\mathbb{T}^n)$ and $\nabla S(x) \neq 0$ on the support of b. Then the following asymptotic formula holds, as $\delta \to 0$,

$$\mathbf{A} f_\delta(x) = \mathbf{a}_0(x, \nabla S(x)) f_\delta(x) + O(\sqrt{\delta}), \tag{27}$$

where the constant in the O-term depends on b and S.

If, specifically, $S(x) = \xi_0 \cdot x$, for some $\xi_0 \in \mathbb{R}^n \backslash \{0\}$, then $O(\sqrt{\delta})$ in formula (27) *can be improved to $O(\delta)$.*

Proof. Formula (27) is a particular case of [37, Theorem 18.1] with the parameters taken $m = \delta = 0$, $N = \rho = 1$ in the notation of [37].

For the second part, let us assume for simplicity that the Fourier transform of $b(x)$ is supported in the ball of radius R. Then we have

$$\begin{aligned} \mathbf{A} f_\delta(x) &= \sum_{k \in \mathbb{Z}^n \backslash \{0\}} \mathbf{a}(x,k) \hat{b}(k - \xi_0 \delta^{-1}) e^{ik\cdot x} \\ &= e^{i\xi_0 \cdot x/\delta} \sum_{|k| \leq R} \mathbf{a}(x, k + \xi_0 \delta^{-1}) \hat{b}(k) e^{ik \cdot x}. \end{aligned}$$

According to our assumptions on $\mathbf{a}$, we obtain

$$\mathbf{a}(x, k + \xi_0 \delta^{-1}) = \mathbf{a}_0(x, \delta k + \xi_0) + \mathbf{a}_1(x, k + \xi_0 \delta^{-1}).$$

Since $|k|$ is bounded and $|\mathbf{a}_1(x,\xi)| \leq C|\xi|^{-1}$, we see that

$$\begin{aligned} \mathbf{a}_0(x, \delta k + \xi_0) &= \mathbf{a}_0(x, \xi_0) + O(\delta), \\ \mathbf{a}_1(x, k + \xi_0 \delta^{-1}) &= O(\delta). \end{aligned}$$

This implies

$$\mathbf{A} f_\delta(x) = \mathbf{a}_0(x,\xi_0) b(x) e^{i\xi_0\cdot x/\delta} + O(\delta). \tag{28}$$

The argument for general b is more technical but similar. □ □

Now we are in a position to use equation (8). We substitute $f(x,t)$ into (8) using (27). Neglecting the terms that vanish as $\delta \to 0$, and canceling the exponent, we obtain

$$b_t + \frac{i}{\delta} b S_t = -(u\cdot\nabla) b - \frac{i}{\delta} b (u\cdot\nabla) S + \mathbf{a}_0(x,\nabla S) b.$$

This yields the following two equations

$$b_t = -(u\cdot\nabla) b + \mathbf{a}_0(x,\nabla S) b \tag{29a}$$

$$S_t = -(u\cdot\nabla) S. \tag{29b}$$

It follows directly from (29b) that the phase is given by

$$S(x,t) = \xi_0 \cdot \varphi_{-t}(x).$$

We take the gradient of (29b) to obtain, with $\xi(x,t) = \nabla S(x,t)$,

$$\xi_t = -(u\cdot\nabla)\xi - \partial u^\top \xi. \tag{30}$$

Rewriting equations (29a) and (30) in the Lagrangian coordinates associated with the flow $x \to x(t) = \varphi_t(x)$ we arrive at a **bicharacteristic-amplitude system (BAS)** of ODE's given by

$$x_t = u(x), \tag{31a}$$

$$\xi_t = -\partial u(x)^\top \xi, \tag{31b}$$

$$b_t = \mathbf{a}_0(x,\xi) b, \tag{31c}$$

subject to initial conditions $x(0) = x_0 \in \mathbb{T}^n$, $\xi(0) = \xi_0 \in \mathbb{R}^n \backslash \{0\}$, $b(0) = b_0 \in \mathbb{C}^d$, and the constraint $b_0 \in F(\xi_0)$.

3.2. **Preservation of constraints.** Let $\mathcal{F}$ be frequency constraints imposed on (8). Then any solution f has to satisfy $\hat{f}(k) \in F(k)$. For solutions of the form (26), in the asymptotic limit $\delta \to 0$, this implies the following condition on the amplitude

$$b(t) \in F(\xi(t)), \quad t \geq 0. \tag{32}$$

This condition however does not automatically hold for solutions of (31c) even if it holds for initial time $t = 0$.

To overcome this deficiency we find a new operator $\mathbf{A}_{\mathrm{new}}$ with principal symbol $\mathbf{a}_{\mathrm{new}}$ which replaces the original $\mathbf{A}$ in the definition of $\mathbf{L}$ such that the action of $\mathbf{L}$ on functions from $H^m_{\mathcal{F}}(\mathbb{T}^n)$ is the same (so

that the group $\mathbf{G}$ remains unchanged on $H^m_{\mathcal{F}}(\mathbb{T}^n)$), while solutions to the new amplitude equation

$$b_t = \mathbf{a}_{\text{new}}(x,\xi)b \tag{33}$$

satisfy (32).

We naturally make use of the identity $\mathbf{L} = \mathbf{\Pi L}$, which holds on functions from $H^m_{\mathcal{F}}(\mathbb{T}^n)$ by the invariance. We notice that $\mathbf{L}$ is a pseudodifferential operator with the symbol

$$-iu(x)\cdot\xi + \mathbf{a}_0(x,\xi) + \mathbf{a}_1(x,\xi).$$

Composing $\mathbf{L}$ with $\mathbf{\Pi}$, which has symbol $\mathbf{p}(\xi)$, we obtain the product of symbols up to $\mathcal{S}^{-1}$

$$-iu(x)\cdot\xi + \tilde{\mathbf{a}}_0(x,\xi) + \mathbf{p}(\xi)\mathbf{a}_0(x,\xi) + \tilde{\mathbf{a}}_1(x,\xi),$$

where $\tilde{\mathbf{a}}_1 \in \mathcal{S}^{-1}$, and $\tilde{\mathbf{a}}_0$ has entries

$$\tilde{\mathbf{a}}_{kl} = -\partial u^\top(x)\xi\cdot\nabla\mathbf{p}_{kl}(\xi), \quad k,l = 1,\dots,d.$$

Notice that for any bicharacteristic curve $(x(t),\xi(t))$, which is a solution of (31a)–(31b), we have

$$\tilde{\mathbf{a}}_{kl}(x(t),\xi(t)) = \frac{d}{dt}\mathbf{p}_{kl}(\xi(t)). \tag{34}$$

So, we obtain the identity $\tilde{\mathbf{a}}_0 = \mathbf{p}_t$, where the time derivative in taken along the bicharacteristics. Let us set

$$\mathbf{a}_{\text{new}}(x,\xi) = \mathbf{p}(\xi)\mathbf{a}_0(x,\xi) + \mathbf{p}_t(\xi). \tag{35}$$

We claim that if this symbol is used in the formulation of the amplitude equation, then $b(t) \in F(\xi(t))$ for all t provided initially $b_0 \in F(\xi_0)$.

Indeed, by (33) and (35), we infer

$$\frac{d}{dt}(\mathbf{id}-\mathbf{p})b = b_t - \mathbf{p}_t b - \mathbf{p}b_t = \mathbf{a}_{\text{new}}b - \mathbf{p}_t b - \mathbf{p}\mathbf{a}_{\text{new}}b = \mathbf{p}\mathbf{p}_t b.$$

From the identity $\mathbf{p} = \mathbf{p}^2$ it follows that $\mathbf{p}_t = \mathbf{p}\mathbf{p}_t + \mathbf{p}_t\mathbf{p}$. Continuing the previous line we obtain

$$\frac{d}{dt}(\mathbf{id}-\mathbf{p})b = \mathbf{p}_t(\mathbf{id}-\mathbf{p})b,$$

and the claim follows from Grönwall's Lemma.

It is easy to check that under the incompressibility constraint given by (16)–(17), transformation (35) takes the form

$$\mathbf{a}_{\text{new}}(x,\xi) = \mathbf{a}_0(x,\xi) + \frac{\xi\otimes\xi}{|\xi|^2}\left(\partial u(x) - \mathbf{a}_0(x,\xi)\right). \tag{36}$$

Remark 3.2. We conclude this section with a general convention for the rest of the paper. If constraints are given, we assume that the symbol has been modified (if necessary) as above so that the BAS preserves the constraints.

3.3. **Examples.** In this section we provide a list of examples of equation (8), which arise from linearizing well-known laws of ideal fluid dynamics. In all examples derivation of the principal symbol can be carried out using the standard calculus of PDO. We illustrate it on the Euler equations.

The Euler equations in velocity form, in any spacial dimension $n = d$, are given by

$$u_t + (u \cdot \nabla)u + \nabla p = 0, \tag{37a}$$

$$\operatorname{div} u = 0. \tag{37b}$$

Let $u(x)$ be a smooth equilibrium solution of (37). The linearized equation takes the form

$$\begin{gathered} f_t = -(u \cdot \nabla)f - (f \cdot \nabla)u - \nabla p, \\ \operatorname{div} f = 0. \end{gathered}$$

Let us rewrite it as follows

$$f_t = -(u \cdot \nabla)f + (f \cdot \nabla)u - 2(f \cdot \nabla)u - \nabla p. \tag{38}$$

The first two terms form the Lie bracket of u and f, which is divergence-free. Therefore, the Leray projection applies only to the third term. So, the pseudodifferential operator $\mathbf{A}$ can be written as

$$\begin{aligned} \mathbf{A}f(x) &= \partial u(x) f(x) - 2 \sum_{k \in \mathbb{Z}^n \setminus \{0\}} \left(\mathbf{id} - \frac{k \otimes k}{|k|^2} \right) (\partial u\, f)\hat{}(k) e^{ik\cdot x} \\ &= \sum_{k \in \mathbb{Z}^n \setminus \{0\}} \left(2\frac{k \otimes k}{|k|^2} - \mathbf{id} \right) (\partial u\, f)\hat{}(k) e^{ik\cdot x}. \end{aligned}$$

We see a composition of two PDOs with symbols $\frac{\xi \otimes \xi}{|\xi|^2}$ and $\partial u(x)$. According to the Composition Formula [37], the symbol of the product is equal to the product of symbols plus a symbol $\mathbf{a}_1(x, \xi)$ of class $\mathcal{S}^{-1}$. Thus, we obtain a decomposition $\mathbf{a} = \mathbf{a}_0 + \mathbf{a}_1$ with the principal part given by

$$\mathbf{a}_0(x, \xi) = \left(2\frac{\xi \otimes \xi}{|\xi|^2} - \mathbf{id} \right) \partial u(x). \tag{39}$$

It is clear that the equation respects the incompressibility constraint.

Similarly, we obtain the following examples.

- *Simple transport, 2D Euler for vorticities, Charney-Hasegawa-Mima* [44]:

$$b_t = 0, \tag{40}$$

- *Euler for velocities* (see [12, 19] and references therein):

$$b_t = \left(2\frac{\xi \otimes \xi}{|\xi|^2} - \mathbf{id}\right) \partial u(x) b. \tag{41}$$

- *Euler for velocities with Coriolis forcing* [43]:

$$b_t = \left(2\frac{\xi \otimes \xi}{|\xi|^2} - \mathbf{id}\right) \partial u(x) b + 2\left(\frac{\xi \otimes \xi}{|\xi|^2} - \mathbf{id}\right) \Omega \times b. \tag{42}$$

- *Euler for vorticities* [23, 24]:

$$b_t = \partial u(x) b - \frac{\omega(x) \cdot \xi}{|\xi|^2} \xi \times b. \tag{43}$$

- *Euler for vorticities with Coriolis forcing* [16]:

$$b_t = \partial u(x) b - \frac{(\omega(x) + 2\Omega) \cdot \xi}{|\xi|^2} \xi \times b. \tag{44}$$

- *Boussinesq approximation* [14]:

$$b_t = \left(2\frac{\xi \otimes \xi}{|\xi|^2} - \mathbf{id}\right) \partial u(x) b + r\left(\mathbf{id} - \frac{\xi \otimes \xi}{|\xi|^2}\right) \nabla \Phi(x), \tag{45a}$$

$$r_t = -b \cdot \nabla \rho_0(x). \tag{45b}$$

- *Camassa-Holm (Euler-α)* [11]:

$$b_t = \left(\frac{\xi \otimes \xi}{|\xi|^2} - \mathbf{id}\right) \partial u^\top(x) b + \frac{\xi \otimes \xi}{|\xi|^2} \partial u(x) b. \tag{46}$$

- *Non-relativistic superconductivity*:

$$b_t = \left(2\frac{\xi \otimes \xi}{|\xi|^2} - \mathbf{id}\right) \partial u(x) b + \left(\mathbf{id} - \frac{\xi \otimes \xi}{|\xi|^2}\right) B \times b. \tag{47}$$

- *Surface quasi-geostrophic equation* [7, 13, 32]:

$$b_t = i\frac{\xi^\perp \cdot \nabla \theta(x)}{|\xi|} b. \tag{48}$$

- *Kinematic dynamo* [2]:

$$b_t = \partial u(x) b. \tag{49}$$

3.4. **BAS as a dynamical system.** The first two equations in (31) form a Hamiltonian system on the symplectic manifold $\Omega^n = T^*\mathbb{T}^n \backslash \{0\}$ with the Hamiltonian

$$H(x,\xi) = u(x)\cdot\xi.$$

We note that on the torus the cotangent bundle is trivial, i.e. $T^*\mathbb{T}^n = \mathbb{T}^n \times \mathbb{R}^n$.

The corresponding phase flow defines a Lebesgue-measure preserving transformation of Ω^n given by

$$\chi_t : (x_0,\xi_0) \to \left(\varphi_t(x_0), \partial\varphi_t^{-\top}(x_0)\xi_0\right), \tag{50}$$

where $\partial\varphi_t(x)$ denotes the Jacobian matrix of φ_t, and $\partial\varphi_t^{-\top}(x)$ is its inverse transpose. In terms of this flow the amplitude equation (31c) can be written as

$$b_t = \mathbf{a}_0(\chi_t(x_0,\xi_0))b. \tag{51}$$

According to our Remark 3.2, (51) defines a dynamical system on bundle $\mathcal{F}$ over Ω^n with fibers $\boldsymbol{\pi}^{-1}(x,\xi) = F(\xi)$. The fundamental solution of (51) defines a smooth linear cocycle over the phase flow χ (see [35])

$$\mathbf{B}_t(x_0,\xi_0) : b_0 \to b(t,x_0,\xi_0,b_0),$$

which maps $F(\xi_0)$ into $F(\xi(t))$. We call it b-cocycle.

Along with the phase flow χ we consider its projectivization, $\overline{\chi}$, onto the compact space $\mathbb{K}^n = \mathbb{T}^n \times \mathbb{S}^{n-1}$, where $\mathbb{S}^{n-1}$ is the unit sphere in $\mathbb{R}^n$. The map $\overline{\chi}_t$ is defined by the rule

$$\overline{\chi}_t : (x_0,\xi_0) \to \left(\varphi_t(x_0), \frac{\partial\varphi_t^{-\top}(x_0)\xi_0}{|\partial\varphi_t^{-\top}(x_0)\xi_0|}\right). \tag{52}$$

Since $\mathbf{a}_0$ is 0-homogenous in ξ, the amplitude equation takes the form

$$b_t = \mathbf{a}_0(\overline{\chi}_t(x_0,\xi_0))b, \tag{53}$$

and hence the b-cocycle can be considered over the compact space $\mathbb{K}^n$ on the projectivized bundle $\mathcal{F}$.

The exponential growth type of $\mathbf{B}$ is defined by the maximal Lyapunov exponent

$$\mu_{\max} = \lim_{t\to\infty}\frac{1}{t}\log \sup_{(x,\xi)\in\mathbb{K}^n}\|\mathbf{B}_t(x,\xi)\|, \tag{54}$$

where the norm is taken over the fiber $F(\xi)$. It is also equal to the largest Lyapunov exponent provided by the Multiplicative Ergodic Theorem for all χ-invariant measures (see [6, Theorem 8.15]).

3.5. **Reduction to L^2 and the $b\xi^m$-cocycle.** In this section we describe a general procedure that will be used to obtain results concerning spectrum on $H^m_{\mathcal{F}}(\mathbb{T}^n)$ automatically from the case $m = 0$. TO this end, we introduce another advective equation

$$f_t = -(u \cdot \nabla) f + \mathbf{A}_m f$$

with the right hand side $\mathbf{L}_m$ on $L^2_{\mathcal{F}}(\mathbb{T}^n)$ being equivalent to the original $\mathbf{L}$ on $H^m_{\mathcal{F}}(\mathbb{T}^n)$ via a similarity relation

$$\mathbf{L}_m = \mathbf{M}_m \mathbf{L} \mathbf{M}_m^{-1}, \tag{55}$$

where $\mathbf{M}_m$ is an isomorphism between $H^m_{\mathcal{F}}(\mathbb{T}^n)$ and $L^2_{\mathcal{F}}(\mathbb{T}^n)$.

Let $\mathbf{M}_m$ be the Fourier multiplier with a smooth scalar non-vanishing symbol equal to $|\xi|^m$ for $|\xi| > 1/2$. Clearly, $\mathbf{M}_m : H^m_{\mathcal{F}}(\mathbb{T}^n) \to L^2_{\mathcal{F}}(\mathbb{T}^n)$ is an isomorphism. Consider the operator $\mathbf{L}_m$ given by (55). By the Composition Formula for PDO, we have

$$\mathbf{L}_m = -(u \cdot \nabla) + \mathbf{A}_m$$

where $\mathbf{A}_m$ is a PDO with principal symbol given by

$$\mathbf{a}_m(x,\xi) = \mathbf{a}_0(x,\xi) - m(\partial u^\top(x)\xi, \xi)|\xi|^{-2}\mathbf{id},$$

for all $|\xi| \geq 1$. The corresponding BAS with the amplitude equation

$$b_t = \mathbf{a}_m(x,\xi) b$$

defines a new cocycle, called $b\xi^m$-cocycle, given by

$$\left(\mathbf{BX}^m\right)_t(x,\xi) = \left|\partial\varphi_t^{-\top}(x)\overline{\xi}\right|^m \mathbf{B}_t(x,\xi), \quad (x,\xi) \in \Omega^n, \tag{56}$$

where $\overline{\xi}$ denotes the unit vector $\xi|\xi|^{-1}$. The $b\xi^m$-cocycle is defined on the same vector bundle $\mathcal{F}$, and is 0-homogenous in ξ. Thus, everything said about the b-cocycle remains valid for the $b\xi^m$-cocycle too.

We note that the similarity relation (55) establishes equivalence of both discrete and essential parts of the spectra. This also concerns the spectra of the corresponding groups. The general procedure will thus be to prove a result in the L^2-space, and deduce the case of arbitrary $m \in \mathbb{R}$ by replacing the b-cocycle with the $b\xi^m$-cocycle. In particular, we will use the maximal Lyapunov exponent of the $b\xi^m$-cocycle defined analogously to (54),

$$\mu^m_{\max} = \lim_{t\to\infty} \frac{1}{t} \log \sup_{(x,\xi)\in\mathbb{K}^n} \| \left(\mathbf{BX}^m\right)_t(x,\xi)\|. \tag{57}$$

4. Essential spectral radius

In this section we establish a formula for the radius of essential spectrum of the semigroup $\mathbf{G}$ on any constrained (or not) Sobolev space.

Theorem 4.1. *Let $\mathbf{G}$ be the C_0-group generated on $H^m_{\mathcal{F}}(\mathbb{T}^n)$, $m \in \mathbb{R}$, by equation (8), in which $u(x)$ is a smooth divergence-free vector field. Then the essential spectral radius of $\mathbf{G}_t$ is given by the formula*

$$r_{\mathrm{ess}}(\mathbf{G}_t) = e^{\mu^m_{\max} t}, \quad t \geq 0. \tag{58}$$

As discussed in Section 3.5 it suffices to prove the theorem only in the case $m = 0$.

The proof consists of two main parts. First, we describe the microlocal structure of the evolution operator $\mathbf{G}_t$. To this end, let us define the following PDO

$$\mathbf{S}_t f(x) = \sum_{k \in \mathbb{Z}^n \backslash \{0\}} \mathbf{B}_t(x,k) \hat{f}(k) e^{ik\cdot x}, \tag{59}$$

and let us put

$$\mathbf{T}_t f = \mathbf{\Pi}[(\mathbf{S}_t f) \circ \varphi_{-t}]. \tag{60}$$

We will prove the following proposition.

Proposition 4.2. *The following decomposition holds for all $t \in \mathbb{R}$*

$$\mathbf{G}_t = \mathbf{T}_t + \mathbf{K}_t, \tag{61}$$

where $\mathbf{K}_t$ is a compact operator on $L^2_{\mathcal{F}}(\mathbb{T}^n)$.

Using Proposition 4.2 and Nussbaum's formula (22) we can rewrite (58) in terms of the operator $\mathbf{T}_t$:

$$r_{\mathrm{ess}}(\mathbf{G}_t) = \lim_{n\to\infty} \|\mathbf{G}_{nt}\|^{1/n}_{\mathcal{C}} = \lim_{n\to\infty} \|\mathbf{T}_{nt}\|^{1/n}_{\mathcal{C}}, \tag{62}$$

where $\mathcal{C}$ is the Calkin algebra over $L^2_{\mathcal{F}}(\mathbb{T}^n)$. We now estimate $\|\mathbf{T}_t\|_{\mathcal{C}}$ in terms of L^∞-norm of the principal symbol of $\mathbf{S}_t$, which is the b-cocycle:

$$\Upsilon(t) = \sup_{(x,\xi)\in\mathbb{K}^n} \|\mathbf{B}_t(x,\xi)\| \tag{63}$$

where as usual the norm in understood over the constraint fiber $F(\xi)$. Notice that by (54) we have $\mu_{\max} = \lim_{t\to\infty} \frac{1}{t} \log \Upsilon(t)$. We will prove the following proposition.

Proposition 4.3. *There exists a constant $C > 0$ such that the following inequalities hold for all $t \in \mathbb{R}$*

$$\Upsilon(t) \leq \|\mathbf{T}_t\|_{\mathcal{C}} \leq C\Upsilon(t). \tag{64}$$

Thus, we obtain

$$\lim_{n\to\infty} \|\mathbf{T}_{nt}\|_{\mathcal{C}}^{1/n} = \lim_{n\to\infty} (\Upsilon(nt))^{1/n} = \left(\lim_{\tau\to\infty} \Upsilon(\tau)^{1/\tau}\right)^t = e^{\mu_{\max} t}.$$

Combining this line with (62) finishes the proof of Theorem 4.1.

Now we prove the above propositions. The proof of Proposition 4.2 uses only basic calculus of PDO (see Shubin [37] or Hörmander [17]). Proposition 4.3 is a consequence of the classical result of Seeley [36] on isomorphism between the algebra of PDO of order 0 modulo compact operators and the algebra of symbols.

of Proposition 4.2. It suffices to consider the case without any constraints. Indeed, if there are constrains imposed on (8), then by "forgetting" about them we can extend $\mathbf{G}$ to all $L^2(\mathbb{T}^n)$. Then by the assumption, (61) holds for the extended group. Applying the projection $\mathbf{\Pi}$ and restricting (61) to $L^2_{\mathcal{F}}(\mathbb{T}^n)$ we obtain (61) in the general case.

So, let us assume $\mathbf{\Pi} \equiv \mathbf{Id}$. Then the operator $\mathbf{T}_t$ takes the form

$$\mathbf{T}_t f = (\mathbf{S}_t f) \circ \varphi_{-t}.$$

By a straightforward computation, we have

$$\frac{d}{dt}\mathbf{T}_t f(x) = -(u(x)\cdot\nabla)\mathbf{T}_t f(x) + \left(\frac{d}{dt}\mathbf{S}_t f\right)(\varphi_{-t}(x)). \tag{65}$$

Using the amplitude equation (31c), we expand the last term in (65) as follows

$$\left(\frac{d}{dt}\mathbf{S}_t f\right)(\varphi_{-t}(x)) = \sum_{k\in\mathbb{Z}^n\setminus\{0\}} \mathbf{a}_0(\varphi_t(y), \partial\varphi_t^{-\top}(y)k)\cdot \\ \cdot \mathbf{B}_t(y,k)\hat{f}(k)e^{ik\cdot y}|_{y=\varphi_{-t}(x)}. \tag{66}$$

Our objective now is to compare this expression with $\mathbf{A}\mathbf{T}_t f$. Let us denote $g = \mathbf{S}_t f$. One has

$$\mathbf{A}\mathbf{T}_t f(x) = \mathbf{A}(g\circ\varphi_{-t}) = (\mathbf{A}' g)\circ\varphi_{-t}, \tag{67}$$

where $\mathbf{A}'$ is a PDO with a semiclassical symbol $\mathbf{a}' \in \mathcal{S}^0$. By the Change of Variables Formula, there is a symbol $\mathbf{a}'_1 \in \mathcal{S}^{-1}$ such that

$$\mathbf{a}'(\varphi_{-t}(x), \xi) = \mathbf{a}_0(x, \partial\varphi_{-t}^{\top}(x)\xi) + \mathbf{a}'_1(x,\xi).$$

Using the identity $\partial\varphi_{-t}^{\top}(x) = \partial\varphi_t^{-\top}(\varphi_{-t}(x))$, we can rewrite the previous as follows

$$\mathbf{a}'(\varphi_{-t}(x), \xi) = \mathbf{a}_0(\varphi_t(y), \partial\varphi_t^{-\top}(y)\xi)|_{y=\varphi_{-t}(x)} + \mathbf{a}'_1(x,\xi). \tag{68}$$

Continuing (67), we obtain

$$
\begin{aligned}
\mathbf{A}\mathbf{T}_t f(x) &= \sum_{k\in\mathbb{Z}^n\setminus\{0\}} \mathbf{a}'(\varphi_{-t}(x),k)\hat{g}(k)e^{ik\cdot\varphi_{-t}(x)} \\
&= \sum_{k\in\mathbb{Z}^n\setminus\{0\}} \mathbf{a}_0(\varphi_t(y),\partial\varphi_t^{-\top}(y)k)\hat{g}(k)e^{ik\cdot y}|_{y=\varphi_{-t}(x)} \qquad (69)\\
&\quad + \mathbf{C}_t^{(1)} f(\varphi_{-t}(x)),
\end{aligned}
$$

where $\mathbf{C}_t^{(1)}$ is a PDO of class $\mathcal{L}^{-1}$. Evidently, the symbol of $\mathbf{C}_t^{(1)}$ is smooth in time.

The principal term in (69) involves a composition of two PDOs with symbols $\mathbf{a}_0(\varphi_t(y),\partial\varphi_t^{-\top}(y)\xi)$ and $\mathbf{B}_t(y,\xi)$, while the right hand side of (66) involves a single PDO with the product of the symbols. By the Composition Formula the difference of (66) and (69) is a PDO $\mathbf{C}_t^{(2)}\in\mathcal{L}^{-1}$ composed with the flow map $\varphi_{-t}(x)$. It also follows from this argument that the symbol of $\mathbf{C}_t^{(2)}$ is smooth in time.

Thus, we have shown that

$$
\left(\frac{d}{dt}\mathbf{S}_t f\right)\circ\varphi_{-t} - \mathbf{A}\mathbf{T}_t f = (\mathbf{C}_t^{(2)} f)\circ\varphi_{-t}.
$$

Going back to (65), we obtain

$$
\frac{d}{dt}\mathbf{T}_t f = -(u\cdot\nabla)\mathbf{T}_t f + \mathbf{A}\mathbf{T}_t f + (\mathbf{C}_t^{(2)} f)\circ\varphi_{-t} = \mathbf{L}\mathbf{T}_t f + (\mathbf{C}_t^{(2)} f)\circ\varphi_{-t}.
$$

By Duhamel's Principle,

$$
\mathbf{T}_t f = \mathbf{G}_t f + \int_0^t \mathbf{G}_{t-s}[(\mathbf{C}_s^{(2)} f)\circ\varphi_{-s}]\,ds. \tag{70}
$$

Let us put

$$
\mathbf{K}_t f = -\int_0^t \mathbf{G}_{t-s}[(\mathbf{C}_s^{(2)} f)\circ\varphi_{-s}]\,ds. \tag{71}
$$

Since the family of operators under the integral is strongly continuous in s and compact, $\mathbf{K}_t$ is a compact operator (the proof of this fact can be found in [10, p. 164]). □ □

of Proposition 4.3. Let us note that in the constraint-free case we have $\|\mathbf{T}_t\|_{\mathcal{C}} = \|\mathbf{S}_t\|_{\mathcal{C}}$. Inequalities (64) then follow from the classical result of Seeley [36] on isomorphism of the subalgebra of PDO's in $\mathcal{C}$ and the space of 0-homogenous symbols.

To extend the result to arbitrary frequency constraints we consider the trivial extension of $\mathbf{B}_t$ to all of $\mathbb{C}^d$ acting by the rule $\mathbf{B}_t^0(x,\xi)b =$

$\mathbf{B}_t(x,\xi)\mathbf{p}(\xi)b$. Let us define the corresponding PDO:

$$\mathbf{S}_t^0 f = \sum_{k\in\mathbb{Z}^n\setminus\{0\}} \mathbf{B}_t^0(x,k)\hat{f}(k)e^{ik\cdot x} : L^2(\mathbb{T}^n) \to L^2(\mathbb{T}^n).$$

According to Seeley, $\|\mathbf{S}_t^0\|_{\mathcal{C}(L^2)} \cong \Upsilon(t)$, because $\|\mathbf{B}_t\| = \|\mathbf{B}_t^0\|$. So, it suffices to show that $\|\mathbf{S}_t^0\|_{\mathcal{C}(L^2)} = \|\mathbf{T}_t\|_{\mathcal{C}(L^2_{\mathcal{F}})}$.

Let us observe that the inequality $\|\mathbf{S}_t^0\|_{\mathcal{C}(L^2)} \geq \|\mathbf{T}_t\|_{\mathcal{C}(L^2_{\mathcal{F}})}$ follows trivially by restriction and projection. To prove the opposite inequality we claim that the operator

$$f \to (\mathbf{Id}-\mathbf{\Pi})[(\mathbf{S}_t^0 f)\circ\varphi_{-t}] : L^2(\mathbb{T}^n) \to L^2(\mathbb{T}^n) \tag{72}$$

is compact.

Indeed, consider the constraint-free extension of $\mathbf{G}$ to $L^2(\mathbb{T}^n)$ as discussed previously. We denote it by $\mathbf{G}^{\text{ext}}$. Let us also denote $\mathbf{B}_t^{\text{ext}}$ the corresponding cocycle obtained as the fundamental solution of (31c) without the constraint condition $b \in F(\xi)$. By Proposition 4.2, we have

$$\mathbf{G}_t^{\text{ext}} = \mathbf{T}_t^{\text{ext}} + \text{compact} \tag{73}$$

on $L^2(\mathbb{T}^n)$, where

$$\mathbf{T}_t^{\text{ext}} = (\mathbf{S}_t^{\text{ext}} f)\circ\varphi_{-t},$$
$$\mathbf{S}_t^{\text{ext}} f = \sum_{k\in\mathbb{Z}^n\setminus\{0\}} \mathbf{B}_t^{\text{ext}}(x,k)\hat{f}(k)e^{ik\cdot x}.$$

Restricting (73) to $L^2_{\mathcal{F}}(\mathbb{T}^n)$ and applying $\mathbf{Id}-\mathbf{\Pi}$ we obtain, by invariance,

$$0 = (\mathbf{Id}-\mathbf{\Pi})[(\mathbf{S}_t f)\circ\varphi_{-t}] + \text{compact} : L^2_{\mathcal{F}} \to L^2.$$

Since $\mathbf{S}_t^0$ is the trivial extension of $\mathbf{S}_t$ it follows that (72) is compact.

Now let us fix any $\varepsilon > 0$, and find a compact operator $\mathbf{K} : L^2_{\mathcal{F}} \to L^2_{\mathcal{F}}$ such that

$$\|\mathbf{T}_t + \mathbf{K}\|_{L^2_{\mathcal{F}}} \leq \|\mathbf{T}_t\|_{\mathcal{C}(L^2_{\mathcal{F}})} + \varepsilon.$$

Let us extend $\mathbf{K}$ to all of L^2 by 0 on the complement of $L^2_{\mathcal{F}}$, and denote the extension by $\mathbf{K}^0$. Then

$$\|\mathbf{\Pi}[\mathbf{S}_t^0(\cdot)\circ\varphi_{-t}] + \mathbf{K}^0\|_{L^2} = \|\mathbf{T}_t + \mathbf{K}\|_{L^2_{\mathcal{F}}}.$$

Writing

$$\mathbf{S}_t^0(\cdot)\circ\varphi_{-t} = \mathbf{\Pi}[\mathbf{S}_t^0(\cdot)\circ\varphi_{-t}] + (\mathbf{Id}-\mathbf{\Pi})[\mathbf{S}_t^0(\cdot)\circ\varphi_{-t}]$$

and using our claim we conclude

$$\begin{aligned}\|\mathbf{S}_t^0\|_{\mathcal{C}(L^2)} &= \|\mathbf{S}_t^0(\cdot)\circ\varphi_{-t}\|_{\mathcal{C}(L^2)} \leq \|\mathbf{\Pi}[\mathbf{S}_t^0(\cdot)\circ\varphi_{-t}] + \mathbf{K}^0\|_{L^2} \\ &= \|\mathbf{T}_t + \mathbf{K}\|_{L^2_{\mathcal{F}}} \leq \|\mathbf{T}_t\|_{\mathcal{C}(L^2_{\mathcal{F}})} + \varepsilon.\end{aligned}$$

This finishes the proof of Proposition 4.3. □ □

4.1. **Shortwave asymptotics.** We can use the explicit representation of the compact term in (61) given by (71) to justify asymptotic formula (26) for the geometric optics solutions. Taking into account the constraints, we consider initially

$$f_\delta = \mathbf{\Pi}[b_0 h_0(x) e^{\xi_0 \cdot x/\delta}], \quad b_0 \in F(\xi_0),\ \delta \ll 1. \tag{74}$$

From (71) we can see that the integral involves a continuous family of pseudodifferential operators of class $\mathcal{L}^{-1}$. Consequently, by Theorem 3.1, $\mathbf{K}_t f_\delta$ decays like $O(\delta)$.

Applying Theorem 3.1 again to the pseudodifferential operator $\mathbf{S}_t$ we obtain the asymptotics of $\mathbf{T}_t f_\delta$. Thus, one has the following formula, as $\delta \to 0$,

$$\mathbf{G}_t f_\delta(x) = \mathbf{B}_t(\varphi_{-t}(x), \xi_0) b_0 h_0(\varphi_{-t}(x)) e^{\xi_0 \cdot \varphi_{-t}(x)/\delta} + O(\delta), \tag{75}$$

where the constant in the O-term depends on t and smoothness of h_0.

One is also interested in the size of time interval on which (75) holds. Sacrificing $O(\delta)$ to a slower term like $O(\sqrt{\delta})$, we can show that (75) holds for all $t \in [0, -c \log \delta]$ with $c > 0$, and O independent of t.

Indeed, the δ-order term in $\mathbf{T}_t f_\delta$ is bounded by the supremum of the ξ-derivative of the b-cocycle. Since the b-cocycle solves the amplitude equation (31c), $\partial_\xi \mathbf{B}_t$ solves

$$\frac{d}{dt} \partial_\xi \mathbf{B}_t = \partial_\xi \mathbf{a}_0 \, \mathbf{B}_t + \mathbf{a}_0 \, \partial_\xi \mathbf{B}_t.$$

Thus, the norm of $\partial_\xi \mathbf{B}_t$ grows at most exponentially. Similar analysis can be made for the δ-order term arising from $\mathbf{K}_t$.

So, we obtain

$$\mathbf{G}_t f_\delta(x) = \mathbf{B}_t(\varphi_{-t}(x), \xi_0) b_0 h_0(\varphi_{-t}(x)) e^{\xi_0 \cdot \varphi_{-t}(x)/\delta} + e^{Ct} O(\delta),$$

for some $C > 0$, and O independent of time. We can choose c to be $(2C)^{-1}$.

4.2. **Unbounded domains.** In the case of unbounded domains, e.g. $\mathbb{R}^n$ or flow channel $\mathbb{R} \times [-L, L]$, the integral (or mixed) analogue of pseudodifferential operators has to be used in the formulation of (8). As we have seen, the proof of Proposition 4.2 uses only basic theorems of pseudodifferential calculus, and those apply for unbounded domains too. Thus, the asymptotic formula (75) remains valid. From it we deduce the lower bound on the radius:

$$r_{\mathrm{ess}}(\mathbf{G}_t) \geq e^{\mu^m_{\max} t}. \tag{76}$$

In the case of the open space $\mathbb{R}^n$ the formula (58) was proved by Vishik in an unpublished version of [45] under the assumption of vanishing velocity at infinity. However, in general the proof of the lower bound breaks down due to non-compactness of PDO from class $\mathcal{L}^{-1}$. In fact, we will show that for the Euler equation on the 2D flow channel $\mathbb{R} \times [-1, 1]$ formula (58) fails.

We use recent results of Z. Lin [28] as our starting point.

Let

$$u(x, y) = \langle U(y), 0\rangle, \quad x \in \mathbb{R},\ y \in [-1, 1],$$

be a steady parallel shear flow with inflectional profile $U(y)$ satisfying the conditions of [28]. It is proved that the eigenvalue problem for the 2D Euler in vorticity formulation

$$\sigma f = \mathbf{L}f = -(u \cdot \nabla)f - (\mathrm{curl}^{-1} f \cdot \nabla)\omega \tag{77}$$

has exact channel wave solutions

$$f = \Delta(\psi(y)e^{i\alpha x}) \tag{78}$$

for all $\sigma \in [0, \sigma_0)$, where $\sigma_0 > 0$. Here ψ is a function from $H^2([-1, 1])$ with $\psi(\pm 1) = 0$.

We denote by X the L^2-space over the channel with periodic boundary conditions on the walls and mean zero condition in the y-direction. On this space curl^{-1} is well-defined, so $\mathbf{L}$ generates a C_0-semigroup.

The normal modes (78) constructed by Z. Lin have infinite energy. In order to put them into X we use a truncation procedure, which replaces the exact identity (77) by approximate identities, turning each σ into an approximate eigenvalue.

Let f be the normal mode (78) satisfying (77). Let γ_N be a smooth function with $\gamma_N(x) = 1$ for $|x| \leq N$, and $\gamma_N(x) = 0$ for $|x| > N + 1$, and $\gamma_N', \gamma_N'', \gamma_N'''$ being uniformly bounded functions. Put

$$f_N = \Delta(\psi(y)\gamma_N(x)e^{i\alpha x}). \tag{79}$$

Then

$$f_N(x, y) = f(x, y)\gamma_N(x) + \psi(y)\gamma_N''(x)e^{i\alpha x} + 2i\alpha\psi(y)\gamma_N'(x)e^{i\alpha x}. \tag{80}$$

Using (80), one obtains the following identity

$$\mathbf{L}f_N = \gamma_N \mathbf{L}f + g_N,$$

where $g_N(x, y)$ is a smooth function supported in $N \leq |x| \leq N+1$ and uniformly bounded in N. On the other hand, from (80) we see that

$$f_N = \gamma_N f + h_N,$$

where h_N possesses similar properties. Thus, we obtain

$$\sigma f_N - \mathbf{L}f_N = \sigma h_N - g_N,$$

and hence,

$$\|\sigma f_N - \mathbf{L} f_N\| \cdot \|f_N\|^{-1} < C\|f_N\|^{-1}.$$

It follows from (80) that $\|f_N\| \sim N^{1/2}$. So, the sequence $\{f_N\|f_N\|^{-1}\}_{N=1}^{\infty}$ is a sequence of approximate eigenfunctions for σ.

This shows that the unstable essential spectrum of the generator, and hence, that of the semigroup, is not empty on X. On the other hand, we can see from (40) that $\mu_{\max} = 0$.

4.3. **Applications to instability.** In this section we indicate several applications of Theorem 4.1 to instability. We recall that a steady state u is linearly unstable if the corresponding semigroup $\mathbf{G}$ is unbounded. A simple sufficient condition for instability of u follows directly from Propositions 4.2, 4.3, and their Sobolev space analogues as explained in Section 3.5. We have

$$\|\mathbf{G}_t\|_{H^m_{\mathcal{F}}(\mathbb{T}^n)} \geq \|\mathbf{G}_t\|_{\mathcal{C}} = \|\mathbf{T}_t\|_{\mathcal{C}} \geq \sup_{(x,\xi)\in\mathbb{K}^n} \|\left(\mathbf{B}\mathbf{X}^m\right)_t(x,\xi)\|. \tag{81}$$

Hence, we obtain the following corollary.

Corollary 4.4. *The steady state $u \in C^\infty(\mathbb{T}^n)$ is unstable in $H^m_{\mathcal{F}}(\mathbb{T}^n)$ if the product*

$$|b(t,x_0,\xi_0,b_0)| \cdot |\xi(t,x_0,\xi_0)|^m$$

is unbounded in $t > 0$ for at least one set of initial data $(x_0,\xi_0) \in \mathbb{K}^n$, $b_0 \in F(\xi_0)$.

An actual unstable mode $f \in H^m_{\mathcal{F}}(\mathbb{T}^n)$ such that $\|\mathbf{G}_t f\| \to \infty$ can be constructed explicitly. We postpone the details of this construction to a later text.

Another consequence of (81) is a sufficient condition for exponential instability in the metric of $H^m_{\mathcal{F}}(\mathbb{T}^n)$, namely, $\mu^m_{\max} > 0$. This condition is satisfied, for instance, by any flow u with exponential stretching of trajectories, provided $|m|$ is sufficiently large. We will show now that for most important equations of type (8) on divergence-free fields, exponential stretching in the flow u implies $\mu_{\max} > 0$, which means instability already in the energy space. Our proof is based on a generalization of the conservation law found by Friedlander and Vishik for the BAS arising from the Euler equation [15].

According to our convention stated in Remark 3.2, we assume that the symbol $\mathbf{a}_0$ has been transformed by the rule (36). One can easily see that every such symbol is invariant with respect to subsequent applications of the transformation (36). This implies the identity

$$(\partial u(x)\,\xi,\xi) = (\mathbf{a}_0(x,\xi)\,\xi,\xi). \tag{82}$$

We use the following notation

$$\langle v_1, v_2, \dots, v_n\rangle = \det[v_1, v_2, \dots, v_n], \tag{83}$$

where the determinant is composed of column-vectors $v_i \in \mathbb{C}^n$.

Theorem 4.5. *Suppose that the BAS* (31) *preserves the incompressibility constraint $b \perp \xi$. Let $b_1, b_2, \dots, b_{n-1}$ be any $n-1$ linearly independent solutions of the amplitude equation over a common initial point (x_0, ξ_0); and let ξ be the corresponding solution of the frequency equation. Then the quantity*

$$\langle b_1, \dots, b_{n-1}, \xi\rangle |\xi|^{-2} \exp\left\{-\int_0^t \operatorname{Tr} \mathbf{a}_0(x(s), \xi(s))ds\right\} \tag{84}$$

is independent of t.

Proof. We start by computing the derivative

$$\begin{aligned} \frac{d}{dt}\langle b_1, \dots, b_{n-1}, \xi\rangle &= \langle \mathbf{a}_0 b_1, \dots, b_{n-1}, \xi\rangle + \dots \\ &\quad + \langle b_1, \dots, \mathbf{a}_0 b_{n-1}, \xi\rangle + \langle b_1, \dots, b_{n-1}, -\partial u^\top \xi\rangle. \end{aligned} \tag{85}$$

We can replace the vector $-\partial u^\top \xi$ in the last determinant without changing it by any other vector that is equal to $-\partial u^\top \xi$ modulo $F(\xi)$. In particular, we can use

$$\partial u^\top \xi - 2\frac{\xi \otimes \xi}{|\xi|^2}\partial u^\top \xi. \tag{86}$$

Furthermore, we can replace the first term $\partial u^\top \xi$ in (86) by $\mathbf{a}_0 \xi$ since their orthogonal projections to the line spanned by ξ are equal, as relation (82) shows. To the second term in (86) we apply the identity

$$-2\frac{\xi \otimes \xi}{|\xi|^2}\partial u^\top \xi = \xi \frac{d}{dt}(\ln |\xi|^2).$$

After these changes we have

$$\langle b_1, \dots, b_{n-1}, -\partial u^\top \xi\rangle = \langle b_1, \dots, b_{n-1}, \mathbf{a}_0 \xi\rangle + \langle b_1, \dots, b_{n-1}, \xi\rangle \frac{d}{dt}(\ln |\xi|^2).$$

Continuing from (85) we obtain

$$\frac{d}{dt}\langle b_1, \dots, b_{n-1}, \xi\rangle = (\operatorname{Tr} \mathbf{a}_0 + \frac{d}{dt}(\ln |\xi|^2))\langle b_1, \dots, b_{n-1}, \xi\rangle.$$

The result now follows by integration. □ □

The traces can be computed directly in all the examples listed in Section 3.3 that are subject to the incompressibility constraint. This yields the following conservation laws.

- Euler for velocities (with or without Coriolis forcing), Camassa-Holm:

$$\langle b_1, \ldots, b_{n-1}, \xi \rangle \equiv const. \tag{87}$$

- 3D Euler for vorticities (with or without Coriolis forcing), kinematic dynamo, superconductivity:

$$\langle b_1, \ldots, b_{n-1}, \xi \rangle |\xi|^{-2} \equiv const. \tag{88}$$

Theorem 4.6. *The equations listed above generate exponentially unstable semigroups on $L^2_{\mathrm{div}}(\mathbb{T}^n)$, provided $u(x)$ has exponential stretching of trajectories.*

Proof. Since $\mathrm{Tr}\, \partial u^\top = 0$, there must exist exponentially growing and exponentially decaying solutions to the ξ-equation (31b).

In the case of (87) we choose a decaying solution $\xi(t)$. By the conservation law, there must exist an exponentially growing solution to the amplitude equation, and hence $\mu_{\max} > 0$.

In the case of (88) we choose a growing solution $\xi(t)$. □ □

5. General inclusion theorem

We continue our discussion with more details on the structure of the essential spectrum. As we see from Theorem 4.1, the maximal Lyapunov exponent of the $b\xi^m$-cocycle contribute a point to the spectrum. In [42] it was observed that any other Lyapunov exponent contributes a point in the same way. In this section we show that, in fact, the entire dynamical spectrum of the $b\xi^m$-cocycle exponentiates into $|\sigma_{\mathrm{ess}}(\mathbf{G}_t)|$ (Theorem 5.3). Under certain aperiodicity assumption on the basic flow φ we prove that points from Σ_m generate circles in $\sigma_{\mathrm{ess}}(\mathbf{G}_t)$ (Theorem 5.4).

Similar results will be obtained for the spectrum of the generator $\mathbf{L}$ (Theorem 5.6). In this case we consider the dynamical spectrum of the cocycle restricted to the submanifold $u(x) \cdot \xi = 0$. This condition has already been used in [20] and its necessity was indicated in [39].

First let us briefly recall general definitions and results from the theory of linear cocycles. Details can be found in [6, 35].

5.1. Cocycles, dynamical spectrum, and Mañe sequences.

Let Θ be a locally compact metric space countable at infinity (such as Ω^n and $\mathbb{K}^n$), and let $\mathcal{E}$ be a finite-dimensional vector bundle over Θ with projection $\boldsymbol{\pi} : \mathcal{E} \to \Theta$. We consider a continuous flow of homeomorphisms on Θ, $\varphi = \{\varphi_t\}_{t\in\mathbb{R}}$, and a linear strongly continuous exponentially bounded cocycle $\boldsymbol{\Phi} = \{\boldsymbol{\Phi}_t(\theta)\}_{t\in\mathbb{R},\theta\in\Theta}$ over φ (a linear extension of φ).

We say that the cocycle $\mathbf{\Phi}$ is **exponentially dichotomic** if there exists a continuous projection-valued function $\mathbf{P}(\theta) : \boldsymbol{\pi}^{-1}(\theta) \to \boldsymbol{\pi}^{-1}(\theta)$, $\theta \in \Theta$, and constants $M > 0$ and $\varepsilon > 0$ such that

1) $\mathbf{P}(\varphi_t(\theta))\mathbf{\Phi}_t(\theta) = \mathbf{\Phi}_t(\theta)\mathbf{P}(\theta)$;
2) $\|\mathbf{\Phi}_t(\theta)\mathbf{P}(\theta)\| \le Me^{-\varepsilon t}$, $t > 0$;
3) $\|\mathbf{\Phi}_t(\theta)(\mathbf{Id} - \mathbf{P})(\theta)\| \le Me^{\varepsilon t}$, $t < 0$.

A point $\lambda \in \mathbb{R}$ is said to belong to the **dynamical** spectrum of $\mathbf{\Phi}$ if the rescaled cocycle $e^{-\lambda t}\mathbf{\Phi}_t$ is not exponentially dichotomic. We denote the dynamical spectrum of $\mathbf{\Phi}$ by $\Sigma_{\mathbf{\Phi}}$. A well-known theorem of Sacker and Sell [35] states that for any cocycle $\mathbf{\Phi}$ over a compact space Θ, its spectrum $\Sigma_{\mathbf{\Phi}}$ consists of the union of a finite number of disjoint intervals

$$\Sigma_{\mathbf{\Phi}} = [r_1^-, r_1^+] \cup \ldots \cup [r_p^-, r_p^+], \tag{89}$$

where the number of intervals p does not exceed dimension of the vector bundle $\mathcal{E}$. The end-points r_1^- and r_p^+ are, respectively, the minimal and the maximal Lyapunov exponents of the cocycle, while all the other Lyapunov exponents (even indexes) belong to $\Sigma_{\mathbf{\Phi}}$, [18].

We now state a characterization of the dynamical spectrum in terms of so-called Mañe sequences. This result can be deduced from works [1, 21], although it has not been explicitly stated. We refer the reader to [38] for an alternative self-contained proof and generalizations to the infinite-dimensional case. First let us recall the notion of a Mañe sequence introduced in [22] (see also [6]).

Definition 5.1. A sequence of pairs $\{(\theta_n, v_n)\}_{n=1}^\infty$, where $\theta_n \in \Theta$ and $v_n \in \boldsymbol{\pi}^{-1}(\theta_n)$, is called a **Mañe sequence** of the cocycle $\mathbf{\Phi}$ if $\{v_n\}_{n=1}^\infty$ is bounded and there are constants $C > 0$ and $c > 0$ such that

$$|\mathbf{\Phi}_n(\theta_n)v_n| > c, \tag{90a}$$

$$|\mathbf{\Phi}_t(\theta_n)v_n| < C, \text{ for all } 0 \le t \le 2n, \tag{90b}$$

for all $n \in \mathbb{N}$.

Theorem 5.2. *For any cocycle $\mathbf{\Phi}$ the following are equivalent:*

(i) $\lambda \in \Sigma_{\mathbf{\Phi}}$;
(ii) *There is a Mañe sequence either for the cocycle $\{e^{-\lambda t}\mathbf{\Phi}_t\}_{t\in\mathbb{R}}$ or its dual.*

Here by dual cocycle we understand the cocycle over the inverse flow φ_{-t} given by $\mathbf{\Phi}_{-t}^{-*}(\theta)$, where $-*$ denotes the inverse of adjoint.

We also recall that if the underlying space Θ is compact, existence of a Mañe sequence is equivalent to existence of a Mañe point introduced

in [29]. The latter is a point $\theta_0 \in \Theta$ for which there exists a (Mañe) vector $v_0 \in \boldsymbol{\pi}^{-1}(\theta_0)$ such that

$$\sup_{t\in\mathbb{R}} |\boldsymbol{\Phi}_t(\theta_0)v_0| < +\infty. \tag{91}$$

We present the proof of this simple fact here as it will be used later in the text.

Let $\{(\theta_n, v_n)\}_{n=1}^{\infty}$ be a Mañe sequence for a cocycle $\boldsymbol{\Phi}$. Since Θ is compact, we may assume that $\varphi_n(\theta_n) \to \theta_0$ and $\boldsymbol{\Phi}_n(\theta_n)v_n \to v_0$. Then, by (90b),

$$|\boldsymbol{\Phi}_t(\theta_0)v_0| = \lim_{n\to\infty} |\boldsymbol{\Phi}_t(\varphi_n(\theta_n))\boldsymbol{\Phi}_n(\theta_n)v_n| = \lim_{n\to\infty} |\boldsymbol{\Phi}_{t+n}(\theta_n)v_n| \le C,$$

for all $t \in \mathbb{R}$.

Conversely, if θ_0 is a Mañe point with Mañe vector v_0, then

$$\theta_n = \varphi_{-n}(\theta_0), \quad v_n = \boldsymbol{\Phi}_{-n}(\theta_0)v_0$$

defines a Mañe sequence.

5.2. **Essential spectrum of the group.** We now present results concerning the essential spectrum of the group $\mathbf{G}$.

We denote by Σ_m the dynamical spectrum of the $b\xi^m$-cocycle. According to the above $\mu^m_{\max}$ is the maximal element of Σ_m, while $\mu^m_{\min}$ will denote the minimal element. If $m = 0$ we simply write Σ, $\mu_{\max}$, $\mu_{\min}$.

Theorem 5.3. *Let $\mathbf{G}$ be the C_0-group generated by equation* (8) *on $H^m_{\mathcal{F}}(\mathbb{T}^n)$, $m \in \mathbb{R}$. Then the following inclusions hold:*

$$\exp\{t\Sigma_m\} \subset |\sigma_{\mathrm{ess}}(\mathbf{G}_t)| \subset \exp\{t[\mu^m_{\min}, \mu^m_{\max}]\}. \tag{92}$$

Proof. According to Section 3.5 we can assume, without loss of generality, that $m = 0$.

In view of Theorem 4.1, we have $|\sigma_{\mathrm{ess}}(\mathbf{G}_t)| \le e^{\mu_{\max} t}$. On the other hand, passing to the inverse operator, we get the identity

$$\sigma_{\mathrm{ess}}(\mathbf{G}_t) = \sigma_{\mathrm{ess}}(\mathbf{G}_{-t})^{-1}. \tag{93}$$

Notice that $\{\mathbf{G}_{-t}\}_{t\in\mathbb{R}}$ is the C_0-group generated by $-\mathbf{L}$. The corresponding amplitude equation is given by

$$b_t = -\mathbf{a}_0(\chi_{-t}(x_0, \xi_0))b.$$

Its solutions define the inverse b-cocycle $\mathbf{B}_{-t}$, whose dynamical spectrum is equal to $-\Sigma$. So, the maximal element in this spectrum is $-\mu_{\min}$. Using Theorem 4.1 we arrive at the formula $r_{\mathrm{ess}}(\mathbf{G}_{-t}) = e^{-\mu_{\min} t}$. In view of (93), this completes the proof of the right inclusion in (92).

Now let $\mu \in \Sigma$. We can assume by rescaling that $\mu = 0$.

Since $0 \in \Sigma$, according to Theorem 5.2 there is a Mañe sequence either for the b-cocycle or for its dual. It is easy to see that the dual b-cocycle arises from the dual group $\mathbf{G}^*$ in the same manner as the b-cocycle arises from $\mathbf{G}$. Since the essential spectra of $\mathbf{G}_t^*$ and $\mathbf{G}_t$ are complex conjugate to each other, there is no loss of generality to assume that there exists a Mañe sequence for the b-cocycle. Let us denote it by $\{(x_n, \xi_n), b_n\}_{n=1}^{\infty}$, where $b_n \in F(\xi_n)$.

We now introduce a two-parameter family of functions. Let $I_{U_n}(x)$ be a smoothed characteristic function of a small open neighborhood U_n containing x_n. Let us define

$$f_{n,\delta}(x) = b_n |U_n|^{-1/2} I_{U_n}(x) e^{i\xi_n \cdot x/\delta},$$
$$g_{n,\delta} = \mathbf{\Pi} f_{n,\delta}.$$

By asymptotic formula (75), we obtain for each $n \in \mathbb{N}$

$$\mathbf{G}_n g_{n,\delta}(x) = \mathbf{B}_n(\varphi_{-n}(x), \xi_n) b_n \frac{I_{U_n}(\varphi_{-n}(x))}{|U_n|^{1/2}} e^{i\xi_n \cdot \varphi_{-n}(x)/\delta} + O(\delta),$$
$$\mathbf{G}_{2n} g_{n,\delta}(x) = \mathbf{B}_{2n}(\varphi_{-2n}(x), \xi_n) b_n \frac{I_{U_n}(\varphi_{-2n}(x))}{|U_n|^{1/2}} e^{i\xi_n \cdot \varphi_{-2n}(x)/\delta} + O(\delta).$$

In view of these identities we can choosing U_n sufficiently small so that

$$\|\mathbf{G}_n g_{n,\delta}\| > c/2 \quad \text{and} \quad \|\mathbf{G}_{2n} g_{n,\delta}\| < 2C, \tag{94}$$

for every $n \in \mathbb{N}$ and $\delta < \delta_n$, where c and C are as in Definition 5.1.

Let us show now that $1 \in |\sigma_{\mathrm{ess}}(\mathbf{G}_t)|$. Indeed, suppose this is not true. Then L^2 admits splitting $L^2 = X_s \oplus X_c \oplus X_u$ into spectral subspaces corresponding to the part the spectrum inside, on, and outside the unit ball, respectively. In addition, X_c is finite-dimensional. Let

$$g_{n,\delta} = g_{n,\delta}^s + g_{n,\delta}^c + g_{n,\delta}^u$$

be the corresponding decomposition of $g_{n,\delta}$. Since $g_{n,\delta} \to 0$ weakly as $\delta \to 0$ for each fixed n, we obtain $\lim_{\delta\to 0} \|g_{n,\delta}^c\| = 0$. Hence, by (94) for sufficiently small δ, we have

$$\|\mathbf{G}_n g'_{n,\delta}\| > c/2 \quad \text{and} \quad \|\mathbf{G}_{2n} g'_{n,\delta}\| < 2C, \tag{95}$$

where $g'_{n,\delta} = g_{n,\delta}^s + g_{n,\delta}^u$. Let us fix a δ for each n so that inequalities (95) hold.

Since the exponential type of $\mathbf{G}_{-t}$ on X_u is negative, there are $\varepsilon > 0$ and $M > 0$ such that $\|\mathbf{G}_t g\| \geq M e^{\varepsilon t} \|g\|$ on X_u. This implies

$$\|\mathbf{G}_{2n} g'_{n,\delta}\| \geq C \|\mathbf{G}_{2n} g_{n,\delta}^u\| \geq C M e^{\varepsilon n} \|\mathbf{G}_n g_{n,\delta}^u\|.$$

Using (95) it follows that $\|\mathbf{G}_n g_{n,\delta}^u\| \leq C_1 e^{-\varepsilon n}$. Then

$$\|\mathbf{G}_n g'_{n,\delta}\| \leq C_2 (\|\mathbf{G}_n g_{n,\delta}^s\| + \|\mathbf{G}_n g_{n,\delta}^u\|) \leq C_3 e^{-\varepsilon n}.$$

This contradicts (95) and hence finishes the proof of the first inclusion in (92). □ □

So far we have imposed no assumption on the basic flow φ. However, the presence of long orbits in φ entails the property of rotational invariance of the spectrum, as we will see from our next result. Similar result for the Mather evolution semigroup generated by the cocycle is well-known (see [6] and references therein). In spite of representation (61), which implies certain resemblance with the Mather semigroup, such a result for the infinite-dimensional group $\mathbf{G}$ is not immediate and requires a separate argument.

In the statement of the theorem we use the concept of a Mañe point. As we noted in Section 5.1 in the case of a compact space Mañe points and Mañe sequences can be used interchangeably. Thus, for the $b\xi^m$-cocycle considered over $\mathbb{K}^n$ both become available.

Theorem 5.4. *Let $\mathbf{G}$ be the C_0-group generated by equation (8) on $H^m_{\mathcal{F}}(\mathbb{T}^n)$, and let $\mu \in \Sigma_m$. Suppose that there exists a point $(x_0, \xi_0) \in \Omega^n$ such that*

(i) *(x_0, ξ_0) is a Mañe point for the rescaled $b\xi^m$-cocycle $e^{-\mu t}\mathbf{B}\mathbf{X}^m_t$, or its dual;*
(ii) *for every given $N > 0$ any open neighborhood of x_0 intersects an φ-orbit of period greater than N.*

Then the following inclusion holds:

$$\mathbb{T} \cdot e^{\mu t} \subset \sigma_{\mathrm{ess}}(\mathbf{G}_t). \tag{96}$$

If every point of the torus satisfies hypothesis (ii), then the flow φ is called **aperiodic**. Thus, from Theorems 5.3 and 5.4 we obtain the following corollary.

Corollary 5.5. *If the flow φ is aperiodic, then the following inclusion holds:*

$$\mathbb{T} \cdot e^{t\Sigma_m} \subset \sigma_{\mathrm{ess}}(\mathbf{G}_t).$$

of Theorem 5.4. In view of the reduction argument presented in Section 3.5 we may assume that $m = 0$.

For definiteness, we assume that there exists a Mañe point (x_0, ξ_0) and Mañe vector $b_0 \in F(\xi_0)$ corresponding to the b-cocycle. In the dual situation we carry out the argument for the adjoint semigroup using the correspondence between $\mathbf{G}^*$ and the dual of $\mathbf{B}$ pointed out in the proof of Theorem 5.3.

Also, by rescaling, we can assume that $\mu = 0$.

We proceed in several steps.

Step 1. *Construction of approximate eigenfunctions.*

We construct a sequence of approximate eigenfunctions for $\mathbf{G}_t$ in the geometric optics form

$$f(x) = b(x)e^{iS(x)/\delta} + O(\delta), \quad \delta \ll 1, \tag{97}$$

with the amplitude $b(x)$ supported in a flow-box stretched along an orbit of large period passing near x_0. We let both b and S propagate along this orbit according to their respective evolution laws defined by the BAS. Several preliminary geometric conditions will have to be settled in order to properly carry out the construction.

First, by the assumption on x_0, for any given $N > 0$ we can choose a point $x_N \in \mathbb{T}^n$ in a vicinity of x_0 so that the period of x_N is greater than $2N + 2$. Second, by taking a small perturbation of ξ_0, if necessary, we can replace ξ_0 by a ξ_N which is not orthogonal to $u(x_N)$. In addition, we choose (x_N, ξ_N) so close to the Mañe point (x_0, ξ_0) that

$$\sup_{-N \le t \le N} |\mathbf{B}_t(x_N, \xi_N)b_0| \le C_1, \tag{98}$$

holds for some $C_1 > 0$ independent of N (see (91)).

These geometric conditions on (x_N, ξ_N) enable us to define a flow-box around the orbit through x_N as follows.

For $\varepsilon > 0$ consider an $(n-1)$-dimensional planar tile Ξ perpendicular to ξ_N, and of spacial measurements $\varepsilon \times \ldots \times \varepsilon$. Choosing ε small enough, every point x in the set $\mathcal{FB}_{\varepsilon,N} = \{\varphi_t(\Xi)\}_{-N \le t \le N}$ is uniquely determined by a $\sigma \in \Xi$ and $t \in [-N, N]$ so that $x = \varphi_t(\sigma)$. This set is the desired flow-box.

For every $\alpha \in [0, 2\pi]$ we will now construct a sequence of functions $f_{\delta,\varepsilon,N}$ in the form (97) such that, for some $C_2 > 0$,

$$\limsup_{\varepsilon \to 0} \limsup_{\delta \to 0} \frac{\|\mathbf{G}_1 f_{\delta,\varepsilon,N} - e^{i\alpha} f_{\delta,\varepsilon,N}\|}{\|f_{\delta,\varepsilon,N}\|} \le \frac{C_2}{N}. \tag{99}$$

Clearly, this is sufficient for proving the lemma.

For every $x \in \mathcal{FB}_{\varepsilon,N}$, $x = \varphi_t(\sigma)$, and $\delta > 0$ we define the amplitude $b(x)$ as follows

$$b(x) = \beta(\sigma)\gamma(t)\mathbf{B}_t(\sigma, \xi_N)b_N, \tag{100}$$

where β is any function on Ξ of unit $L^2(\Xi)$-norm, and where γ is a slowly varying tent-shaped function defined as $\gamma(t) = 1 - |t|N^{-1}$, for $-N \le t \le N$, and $\gamma(t) = 0$ otherwise.

Let us define a phase by the rule

$$S(x)|_{x=\varphi_t(\sigma)} = t. \tag{101}$$

Observe that $\nabla S|_{\Xi}$ is proportional to ξ_N, and $S(\varphi_t(x)) - S(x) = t$ for all x in the flow-box. Taking the gradient at $x = \sigma$ we obtain

$$\begin{aligned} \partial\varphi_t^{\top}(\sigma)\nabla S(\varphi_t(\sigma)) &= \nabla S(\sigma) \\ \nabla S(\varphi_t(\sigma)) &= \partial\varphi_t^{-\top}(\sigma)\nabla S(\sigma). \end{aligned}$$

So, up to a constant multiple,

$$S(x)|_{x=\varphi_t(\sigma)} = \partial\varphi_t^{-\top}(\sigma)\xi_N, \tag{102}$$

for all $x \in \mathcal{FB}_{\varepsilon,N}$. Notice that $\nabla S(x) \neq 0$ on the flow-box.

Now, we put

$$f_{\delta,\varepsilon,N} = \mathbf{\Pi}[be^{iS/\delta}]. \tag{103}$$

By Theorem 3.1, we conclude that $f_{\delta,\varepsilon,N}$ is of the form (97).

Step 2. *Shortwave asymptotics.*

Let us apply $\mathbf{G}_1$ to $f = f_{\delta,\varepsilon,N}$. The action of $\mathbf{G}_1$ on f with fixed ε and N, in the asymptotic limit $\delta \to 0$, is easily found using routine application of Proposition 4.2, the Change of Variables Formula for pseudodifferential operators, and Theorem 3.1 applied to the operator $\mathbf{S}_t$ defined by (59). As a result, one obtains the following formula

$$\mathbf{G}_1 f(x) = \mathbf{B}_1(\varphi_{-1}(x), \nabla S(\varphi_{-1}(x))) f(\varphi_{-1}(x)) + o(1),$$

as $\delta \to 0$. So, if $x = \varphi_t(\sigma)$, then by (102) we obtain, up to the leading order term,

$$\begin{aligned} \mathbf{G}_1 f(x) &= \beta(\sigma)\gamma(t-1)\mathbf{B}_1(\varphi_{t-1}(\sigma), \partial\varphi_{t-1}^{-\top}(\sigma)\xi_N)\mathbf{B}_{t-1}(\sigma,\xi_N)b_0 e^{i(t-1)/\delta} \\ &= e^{-i/\delta}\beta(\sigma)\gamma(t-1)\mathbf{B}_t(\sigma,\xi_N)b_0 e^{it/\delta} \\ &= e^{-i/\delta} f(x) + e^{-i/\delta}\beta(\sigma)(\gamma(t-1) - \gamma(t))\mathbf{B}_t(\sigma,\xi_N)b_0 e^{it/\delta}. \end{aligned}$$

Let us take δ of the form $(2\pi k - \alpha)^{-1}$, $k \in \mathbb{N}$. Then from the above we conclude

$$\mathbf{G}_1 f - e^{i\alpha} f = e^{i\alpha}\beta(\sigma)(\gamma(t-1) - \gamma(t))\mathbf{B}_t(\sigma,\xi_N)b_0 e^{it/\delta} + o(1).$$

It is readily seen that the lim sup of the energy norm of the left hand side, as $\delta \to 0$, is bounded by the energy norm of

$$\beta(\sigma)(\gamma(t-1) - \gamma(t))\mathbf{B}_t(\sigma,\xi_N)b_0.$$

Step 3. *Change of variables over the flow-box.*

When performing integration over the flow-box, it is convenient to switch from x- to (σ, t)-variables.

To this end, we define a map from $\mathcal{S} = [-N, N] \times \Xi$ onto $\mathcal{FB}_{\varepsilon,N}$ by

$$H(t,\sigma) = \varphi_t(\sigma).$$

A direct computation shows that

$$\partial H(t,\sigma) = \left[u \circ \varphi_t(\sigma),\, \partial_{\sigma_1}\varphi_t(\sigma), \ldots,\, \partial_{\sigma_{n-1}}\varphi_t(\sigma)\right],$$

where $\sigma = (\sigma_1, \ldots, \sigma_{n-1})$ is a system of rectangular coordinates on Ξ. Let e_k be the unit vector in the σ_k-th direction. Then

$$\partial H(t,\sigma) = \partial\varphi_t(\sigma)\,[u(\sigma),\, e_1, \ldots,\, e_{n-1}].$$

Consequently, the quantity

$$\kappa(\sigma) = |\det \partial H(t,\sigma)| = |\det [u(\sigma),\, e_1, \ldots,\, e_{n-1}]|,$$

is independent of t. Besides, $\kappa(0) \neq 0$ due to our assumption that $\xi_N \not\perp u(x_N)$.

Step 4. *Proof of* (99). We obtain

$$\limsup_{\delta\to 0} \frac{\|\mathbf{G}_1 f - e^{i\alpha} f\|^2}{\|f\|^2} \le \frac{I_1}{I_2},$$

where in the (σ, t)-coordinates,

$$I_1 = \int_{\mathcal{S}} \kappa(\sigma)\beta^2(\sigma)(\gamma(t-1) - \gamma(t))^2 |\mathbf{B}_t(\sigma, \xi_N) b_0|^2 d\sigma dt,$$
$$I_2 = \int_{\mathcal{S}} \kappa(\sigma)\beta^2(\sigma)\gamma^2(t) |\mathbf{B}_t(\sigma, \xi_N) b_0|^2 d\sigma dt.$$

Now, let us shrink the tile Ξ to the point x_N – i.e. let $\varepsilon \to 0$. Then $\beta(\sigma)$ serves as an approximative kernel. We obtain

$$I_1 \to \kappa(0) \int_{-N}^{N} (\gamma(t-1) - \gamma(t))^2 |\mathbf{B}_t(x_N, \xi_N) b_0|^2 dt,$$
$$I_2 \to \kappa(0) \int_{-N}^{N} \gamma^2(t) |\mathbf{B}_t(x_N, \xi_N) b_0|^2 dt.$$

Notice that $|\gamma(t-1) - \gamma(t)| \le N^{-1}$ by construction. Thus, using the previous identities and (98), we estimate

$$\limsup_{\varepsilon\to 0} \frac{I_1}{I_2} \lesssim \frac{N^{-2} \int_{-N}^{N} |\mathbf{B}_t(x_N, \xi_N) b_0|^2 dt}{|b_0|^2} \le \frac{C_2}{N}.$$

This finishes the proof of the theorem. □ □

5.3. **Essential spectrum of the generator.** For general advective equations (8) the spectral mapping theorem is unknown. However, we can obtain similar results about the essential spectrum of $\mathbf{L}$ considering the dynamical spectrum of the $b\xi^m$-cocycle restricted to the χ-invariant submanifold

$$\Omega_0^n = \{(x, \xi) \in \Omega^n : u(x) \cdot \xi = 0\},$$

which is also 0-homogenous in ξ, and projects onto a submanifold of $\mathbb{K}^n$. Let us denote the spectrum of the cocycle restricted to Ω_0^n by $\Sigma_m^\perp$.

We show below that the analogues of Theorems 5.3 and 5.4 hold for $\mathbf{L}$ if the full dynamical spectrum is replaced by $\Sigma_m^\perp$.

Theorem 5.6. *Let $\mathbf{L}$ be the operator defined by equation (8) on $H^m_{\mathcal{F}}(\mathbb{T}^n)$, $m \in \mathbb{R}$. Then the following inclusions hold:*

$$\Sigma_m^\perp \subset \operatorname{Re}\sigma_{\text{ess}}(\mathbf{L}) \subset [\mu^m_{\min}, \mu^m_{\max}]. \tag{104}$$

Furthermore, if there is a Mañe point $(x_0, \xi_0) \in \Omega_0^n$ corresponding to μ satisfying the assumptions (i) and (ii) of Theorem 5.4, then

$$\mu + i\mathbb{R} \subset \sigma_{\text{ess}}(\mathbf{L}). \tag{105}$$

In particular, if the flow φ is aperiodic, then one has

$$\Sigma_m^\perp + i\mathbb{R} \subset \sigma_{\text{ess}}(\mathbf{L}). \tag{106}$$

Proof. We first notice that the right inclusion follows immediately from the general inclusion for essential spectra (24), and Theorem 5.3.

Now let $\mu \in \Sigma_m^\perp$. As before, we may assume that $\mu = 0$, $m = 0$, and that there exists a Mañe point $(x_0, \xi_0) \in \Omega_0^n$ for the b-cocycle. We consider three cases, which we call aperiodic, periodic, and the case of stagnation point.

Case 1: aperiodic.

Suppose x_0 is aperiodic, i.e. assumption (ii) of Theorem 5.4 holds. We then aim at proving (105), which in particular implies (104).

We define

$$f_{\delta,\varepsilon,N} = \mathbf{\Pi}[b e^{iS/\delta} e^{i\alpha t}],$$

where all the ingredients are the same as in the proof of Theorem 5.4 except for the phase function. We define S as follows.

Let Ξ be a planar $(n-1)$-dimensional tile orthogonal to $u(x_0)$, containing x_0, and having spacial measurements $\varepsilon \times \ldots \times \varepsilon$. Since $\xi_0 \perp u(x_0)$, $\xi_0 \in \Xi$. Let $\tilde{\Xi} \subset \Xi$ be the orthogonal complement of ξ_0 in Ξ containing x_0. So, the surface $\{\varphi_t(\tilde{\Xi})\}_{-N \le t \le N}$ is orthogonal to ξ_0 at x_0.

In the flow-box, defined by $\mathcal{FB}_{\varepsilon,N} = \{\varphi_t(\Xi)\}_{-N \le t \le N}$, we introduce the following coordinates

$$x \to (\tilde{\sigma}, \tau, t),$$
$$x = \varphi_t(\sigma), \quad \sigma = \tilde{\sigma} + \tau\xi_0.$$

Using these coordinates let us define the phase as follows:

$$S(x)|_{x=\varphi_t(\tilde{\sigma}+\tau\xi_0)} = \tau. \tag{107}$$

Then $S(x) = S(\varphi_t(x))$ for all $x \in \mathcal{FB}_{\varepsilon,N}$. This implies that $\nabla S(x)$ solves the ξ-equation, and by construction,

$$u(x) \cdot \nabla S(x) = 0. \tag{108}$$

Using (108) and Theorem 3.1 (notice that $\nabla S(x) \neq 0$ in the flow-box!), we obtain, as $\delta \to 0$,

$$\mathbf{L}f - i\alpha f = \beta(\sigma)\gamma'(t)\mathbf{B}_t(\sigma, \xi_0)b_0 e^{i\alpha t}e^{i\tau/\delta} + o(1).$$

The rest of the proof goes along the lines of Theorem 5.4.

Case 2: periodic.

In this case we assume $u(x_0) \neq 0$, and there is a $P > 0$ and an open neighborhood of x_0, denoted U_{x_0}, such that $p(x) < P$ for all $x \in U_{x_0}$, where $p(x)$ denotes the prime period of x. Let us also define the continuous period function

$$p_c(x) = \lim_{\varepsilon \to 0} \sup_{|x-y|<\varepsilon} p(y). \tag{109}$$

For any small $\varepsilon > 0$, let Ξ be the planar $(n-1)$-dimensional tile of spacial measurements $\varepsilon \times \ldots \times \varepsilon$, orthogonal to $u(x_0)$. Due to the periodicity assumption, the flow-box, defined by $\mathcal{FB}_\varepsilon = \{\varphi_t(\Xi)\}_{t\in\mathbb{R}}$, has the shape of the torus. We define the phase $S(x)$ by (107) as before. Since $S(x)$ is flow invariant, it is well-defined in $\mathcal{FB}_\varepsilon$.

Our further argument is based on the following claim.

Claim 5.7. One has $\partial\varphi^{-\top}_{p_c(x_0)}(x_0)\xi_0 = \xi_0$. So, the flow χ is $p_c(x_0)$-periodic at (x_0, ξ_0).

Proof. Notice that $p_c(x)$ is an integer multiple of $p(x)$, and p_c is a continuous function where it is finite. We have $\varphi_{p_c(x)}(x) = x$ for all x in an open neighborhood of x_0. By the Implicit Function Theorem, $p_c(x)$ is differentiable at x_0 and

$$\partial\varphi_{p_c(x_0)}(x_0) + u(x_0) \otimes \nabla p_c(x_0) = \mathbf{id}$$

Hence,

$$\partial\varphi_{p_c(x_0)}(x_0) = \mathbf{id} - u(x_0) \otimes \nabla p_c(x_0).$$

One has

$$\partial\varphi^{-\top}_{p_c(x_0)}(x_0) = \partial\varphi^{\top}_{-p_c(x_0)}(x_0) = \mathbf{id} + \nabla p_c(x_0) \otimes u(x_0).$$

Since $\xi_0 \perp u(x_0)$, then clearly, $\partial\varphi^{-\top}_{p_c(x_0)}(x_0)\xi_0 = \xi_0$. □ □

Since (x_0, ξ_0) is a Mañe point for the b-cocycle, there exists a vector $b_0 \in F(\xi_0)$ such that the sequence

$$\{\mathbf{B}^k_{p_c(x_0)}(x_0, \xi_0)b_0\}_{k\in\mathbb{Z}}$$

is bounded. This implies that there is a $\lambda(x_0) \in \mathbb{T}$ and $b(x_0) \in F(\xi_0)$ such that

$$\mathbf{B}_{p_c(x_0)}(x_0, \xi_0) b(x_0) = \lambda(x_0) b(x_0).$$

By continuity, for every $\sigma \in \Xi$ there exists a $b(\sigma) \in F(\nabla S(\sigma))$ and $\lambda(\sigma) \in \mathbb{C}$ such that the following holds:

$$\mathbf{B}_{p_c(\sigma)}(\sigma, \nabla S(\sigma)) b(\sigma) = \lambda(\sigma) b(\sigma), \tag{110}$$

$$|\lambda(\sigma) - \lambda(x_0)| = o(1), \text{ as } \varepsilon \to 0. \tag{111}$$

Now we define the function $f_{\delta,\varepsilon}$ as follows:

$$f_{\delta,\varepsilon} = \mathbf{\Pi}[b e^{iS/\delta}],$$

where the amplitude is given by

$$b(x) = \beta(\sigma) \lambda(\sigma)^{-t/p_c(\sigma)} \mathbf{B}_t(\sigma, \nabla S(\sigma)) b(\sigma), \quad x \in \mathcal{FB}_\varepsilon,$$

and where by $\lambda(\sigma)^{-t/p_c(\sigma)}$ we understand the principal branch of the power function.

This amplitude function $b(x)$ is well-defined in the flow-box only if the equality

$$p_c(\sigma) = p(\sigma) \tag{112}$$

holds for all $\sigma \in \Xi$. It is easy to check that $p(x)$ is a lower-semicontinuous function, and, as any such function, it is continuous on a dense G_δ-set. Thus, the set $A = \{x : p_c(x) = p(x)\}$ is dense and, evidently, φ-invariant in $\mathbb{T}^n$. Moreover, as we noted above, $p_c(x)$ is an integer multiple of $p(x)$. So, if $x \in A$ and $0 < p(x) < \infty$, then an open neighborhood of x belongs to A.

By virtue of the periodicity assumption, we have $0 < p(\sigma) < P$ for all $\sigma \in \Xi$. Hence, A has a non-empty intersection with Ξ. In order to ensure that (112) holds on Ξ it suffices to restrict Ξ to a smaller tile contained in A. In the sequel, Ξ denotes such a restriction.

Now, since the definition of $f_{\delta,\varepsilon}$ is validated, we show that $\{f_{\delta,\varepsilon}\}$ is a sequence of approximate eigenfunctions corresponding to the point

$$z = i \arg \lambda(x_0)/p_c(x_0).$$

Indeed, routine computations, based on an application of Theorem 3.1 and the fact that the b-cocycle solves (31c), reveal the following asymptotic formula for the action of $\mathbf{L}$ on $f_{\delta,\varepsilon}$ at $x = \varphi_t(\sigma)$:

$$\begin{aligned}\mathbf{L} f_{\delta,\varepsilon} &= -(u \cdot \nabla) f_{\delta,\varepsilon} + \mathbf{A} f_{\delta,\varepsilon} \\ &= \ln(\lambda(\sigma)) p^{-1}(\sigma) \beta(\sigma) \lambda(\sigma)^{-t/p_c(\sigma)} \mathbf{B}_t(\sigma, \nabla S(\sigma)) b(\sigma) e^{iS/\delta} \\ &\qquad - \mathbf{a}_0(x, \nabla S(x)) f_{\delta,\varepsilon} + \mathbf{a}_0(x, \nabla S(x)) f_{\delta,\varepsilon} + o(1),\end{aligned}$$

as $\delta \to 0$ for each fixed $\varepsilon > 0$. So,

$$\mathbf{L} f_{\delta,\varepsilon} - z f_{\delta,\varepsilon} = \ln |\lambda(\sigma)| p^{-1}(\sigma) f_{\delta,\varepsilon} + o(1).$$

By (111), the logarithm is arbitrarily small, as $\varepsilon \to 0$. Thus, letting $\delta \to 0$ first, then letting $\varepsilon \to 0$ completes the proof in the periodic case.

CASE 3: STAGNATION POINT.

In the case of stagnation point we have $u(x_0) = 0$ and still $p(x) < P$ for all $x \in U_{x_0}$.

First of all, we single out a simple situation when there is an open neighborhood of x_0, denoted V_{x_0}, consisting entirely of stagnation points of the flow φ. In this case we let S be any function such that $\nabla S(x_0) = \xi_0$. For $\varepsilon > 0$ pick a function $h_\varepsilon(x)$ of unit L^2-norm supported in $\{|x - x_0| < \varepsilon\}$ so that for sufficiently small ε the support of h_ε is concentrated inside V_{x_0}.

Since $\{\mathbf{B}_t(x_0, \xi_0) b_0\}_{t \in \mathbb{R}}$ is bounded, the matrix $\mathbf{a}_0(x_0, \xi_0)$ has a purely imaginary eigenvalue $i\alpha$. Let v_0 be the corresponding eigenvector. We set

$$f_{\delta,\varepsilon} = \mathbf{\Pi}[v_0 h_\varepsilon e^{iS/\delta}].$$

Then

$$\begin{aligned} \mathbf{L} f_{\delta,\varepsilon} - i\alpha f_{\delta,\varepsilon} &= \mathbf{a}_0(x, \nabla S(x)) v_0 h_\varepsilon(x) e^{iS(x)/\delta} - i\alpha v_0 h_\varepsilon(x) e^{iS(x)/\delta} + o(1) \\ &= (\mathbf{a}_0(x, \nabla S(x)) - i\alpha) v_0 h_\varepsilon(x) e^{iS(x)/\delta} + o(1). \end{aligned}$$

As before, we let $\delta \to 0$ first, and then $\varepsilon \to 0$.

Now, suppose that every neighborhood of x_0 contains a non-stagnant point. Since the periods in a vicinity of x_0 are bounded, x_0 is Lyapunov stable. This implies that for every $\varepsilon > 0$ there is a proper orbit $\mathcal{O}_\varepsilon$ contained entirely in $\{|x - x_0| < \varepsilon\}$. By the density and invariance of A we can ensure the identity (112) on the orbit $\mathcal{O}_\varepsilon$. Let p_ε denote the prime period of $\mathcal{O}_\varepsilon$.

Since the orbit $\mathcal{O}_\varepsilon$ is contained in a small neighborhood of x_0, we have

$$\int_{\mathcal{O}_\varepsilon} u(x) \cdot \xi_0 \; |dx| = 0.$$

Hence, there is a point $x_\varepsilon \in \mathcal{O}_\varepsilon$ such that $u(x_\varepsilon) \cdot \xi_0 = 0$.

Next, for each $\varepsilon > 0$ we find a natural N_ε such that

$$P/3 < N_\varepsilon p_\varepsilon < P. \tag{113}$$

As before, the matrix $\mathbf{B}_1(x_0, \xi_0)$ has an eigenvalue λ_0 with $|\lambda_0| = 1$. So, by perturbation, the matrix $\mathbf{B}_{N_\varepsilon p_\varepsilon}(x_\varepsilon, \xi_0) = \mathbf{B}_{p_\varepsilon}^{N_\varepsilon}(x_\varepsilon, \xi_0)$ has an eigenvalue λ_ε such that

$$|\lambda_\varepsilon - \lambda_0^{N_\varepsilon p_\varepsilon}| = o(1), \text{ as } \varepsilon \to 0. \tag{114}$$

Observe that $\lambda_\varepsilon^{1/N_\varepsilon}$ is an eigenvalue of $\mathbf{B}_{p_\varepsilon}(x_\varepsilon, \xi_0)$. So, using the result of the previous periodic case, we find

$$p_\varepsilon^{-1} \ln(\lambda_\varepsilon^{1/N_\varepsilon}) \in \sigma(\mathbf{L}).$$

On the other hand,

$$p_\varepsilon^{-1} \ln(\lambda_\varepsilon^{1/N_\varepsilon}) = (N_\varepsilon p_\varepsilon)^{-1}(\ln |\lambda_\varepsilon| + i \arg \lambda_\varepsilon).$$

By (113) and (114), this sequence of spectral points is bounded and at the same time the real parts vanish as $\varepsilon \to 0$. So, there is a subsequence converging to a purely imaginary point.

Since the approximate eigenfunctions used in the proof are weakly-null, the found point lies in the essential spectrum of the generator.

This completes the proof of (104). □ □

6. Spectrum in Sobolev spaces $H^m_{\mathcal{F}}(\mathbb{T}^n)$ for large $|m|$

In this section we describe the results concerning structure of the essential spectrum over Sobolev spaces of sufficiently large smoothness in the case when the basic flow has a nonzero Lyapunov exponent.

First, we seek sufficient conditions for the spectrum Σ_m to be connected, i.e.

$$\Sigma_m = [\mu^m_{\min}, \mu^m_{\max}]. \tag{115}$$

By Theorem 5.3, from (115) we immediately obtain the identity

$$|\sigma_{\mathrm{ess}}(\mathbf{G}_t)| = \exp\{t[\mu^m_{\min}, \mu^m_{\max}]\}. \tag{116}$$

One trivial condition that guarantees (115) follows from Sacker and Sell's theorem stated in Section 5.1. Namely, the dimension of the vector bundle $\mathcal{F}$ is one.

Second, we identify certain margins of the spectrum, $[\mu^m_{\min}, s] \cup [S, \mu^m_{\max}]$, which will be proved to satisfy the aperiodicity assumption (ii) of Theorem 5.4. A condition on m will be found that insures that these margins are nonempty, and the same condition will imply (115). Hence, from Theorem 5.4 and (116) we conclude that $\sigma_{\mathrm{ess}}(\mathbf{G}_t)$ contains solid spectral rings and has no circular gaps. A generic configuration of such spectrum is indicated in Figure 1.

Third, we establish similar results for $\sigma_{\mathrm{ess}}(\mathbf{L})$, and under the same condition on m we show that $\Sigma^\perp_m = \Sigma_m$. A generic spectrum in this case is shown in Figure 2. Consequently, we obtain a variant of the spectral mapping property for the group $\mathbf{G}$, which in turn implies the identity between the exponential type of $\mathbf{G}$ and the spectral bound of $\mathbf{L}$.

Before we state our results, let us introduce relevant notation.

Let $\lambda_{\min}$ and $\lambda_{\max}$ denote the end-points of the dynamical spectrum of the ξ-cocycle $\partial\varphi_t^{-\top}$. As we noted earlier, this cocycle is the fundamental matrix solution of the ξ-equation (31b). Since $\partial\varphi_t^{-\top}$ is the inverse dual to the Jacobi cocycle $\partial\varphi_t$ (see Section 5.1), the end-points of the latter are $\ell_{\min} = -\lambda_{\max}$, $\ell_{\max} = -\lambda_{\min}$. In view of the incompressibility assumption $\det(\partial\varphi_t) = 1$, the conditions $\lambda_{\max} > 0$ and $\ell_{\max} > 0$ are equivalent. If either of them holds we say that φ has **exponential stretching of trajectories**. From now on we only use the exponents $\lambda_{\max}$ and $\lambda_{\min}$.

Let us introduce the following constants

$$s = \sup_{k\in\mathbb{R}}\{\mu^k_{\min}\}, \quad S = \inf_{k\in\mathbb{R}}\{\mu^k_{\max}\}. \tag{117}$$

As we will see, any point of Σ_m that lies outside the interval $[s,S]$ (if such a point exists) satisfies the aperiodicity assumption of Theorem 5.4.

Our main result is stated in the following theorem.

Theorem 6.1. *Let* $\mathbf{G}$ *be the* C_0*-group generated by equation* (8) *on* $H^m_{\mathcal{F}}(\mathbb{T}^n)$*,* $m \in \mathbb{R}$*. Assume that* $\lambda_{\max} > 0$ *and* $|m| > \frac{\mu_{\max}-\mu_{\min}}{\lambda_{\max}-\lambda_{\min}}$*. Then the following holds:*

1) *identities* (115) *and* (116) *;*
2) $\mu^m_{\min} < s$ *and* $S < \mu^m_{\max}$ *;*
3) $\mathbb{T}\cdot\exp\{t[\mu^m_{\min}, s]\cup[S,\mu^m_{\max}]\} \subset \sigma_{\text{ess}}(\mathbf{G}_t)$ *;*

Thus, a generic spectral picture is reminiscent of the bicycle wheel as shown in Figure 1. The crucial features of the spectrum in this case are two solid inner and outer rings, and no circular spectral gap.

Similar description can be given to the spectrum of the generator.

Theorem 6.2. *Under the assumption of Theorem 6.1 the following holds:*

1) $\Sigma^\perp_m = \Sigma_m = [\mu^m_{\min}, \mu^m_{\max}]$ *;*
2) $[\mu^m_{\min}, s]\cup[S,\mu^m_{\max}] + i\mathbb{R} \subset \sigma_{\text{ess}}(\mathbf{L})$ *;*
3) $\operatorname{Re}\sigma_{\text{ess}}(\mathbf{L}) = [\mu^m_{\min}, \mu^m_{\max}]$.

A generic spectral picture in this case is shown in Figure 2. Results of [25] show that the ladder structure is possible for elliptic flows, even though those have no exponential stretching.

Combining the above theorems with (23) we obtain the following spectral mapping property.

Theorem 6.3 (Annular Hull Theorem)**.** *Under the assumptions of Theorem 6.1 one has*

$$\mathbb{T}\cdot\sigma(\mathbf{G}_t) = \exp\{t\sigma(\mathbf{L}) + i\mathbb{R}\}. \tag{118}$$

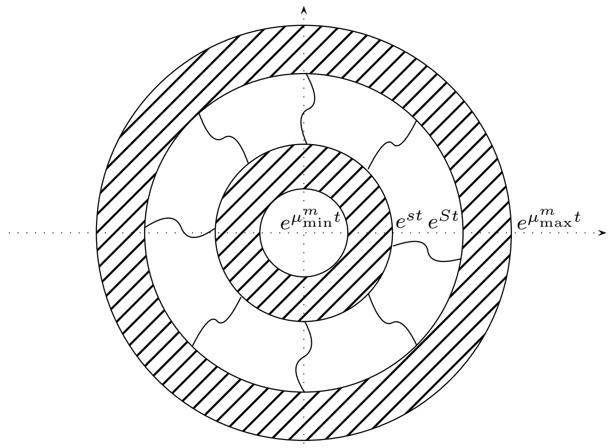

FIGURE 1. "Bicycle wheel" structure of the essential spectrum of $\mathbf{G}_t$ over $H^m_{\mathcal{F}}(\mathbb{T}^n)$, under the assumptions of Theorem 6.1.

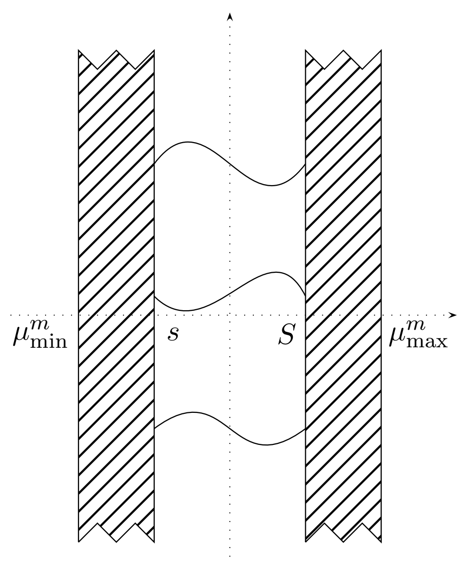

FIGURE 2. "Ladder" structure of the essential spectrum of $\mathbf{L}$ over $H^m_{\mathcal{F}}(\mathbb{T}^n)$ under the assumptions of Theorem 5.6.

As a consequence, we obtain the identity between the exponential type of the semigroup, and the spectral bound of the generator:

$$\omega(\mathbf{G}) = s(\mathbf{L}). \tag{119}$$

The proofs of Theorems 6.1 and 6.2 given in the next section will come out as a result of systematic study of the dynamical spectrum Σ_m.

Let us consider now one particular case when $\Sigma = \{0\}$, i.e. $\mu_{\max} = \mu_{\min} = 0$. So, the amplitude equation has no exponentially growing or decaying solutions. Examples from our list that trivially satisfy this condition are the 2D Euler in vorticity formulation, simple transport, SQG, and CHM equations. In this case the hypothesis of Theorem 6.1 is satisfied for all $m \neq 0$ provided $\lambda_{\max} > 0$. Thus, $S \leq 0 \leq s$, and from Theorems 6.1 and 6.2 we obtain the following corollary.

Corollary 6.4. *Suppose that $\lambda_{\max} > 0$ and $\Sigma = \{0\}$. Then for any $m \neq 0$ one has the identities*

$$\sigma_{\mathrm{ess}}(\mathbf{G}_t) = \mathbb{T} \cdot \exp\{tm[\lambda_{\min}, \lambda_{\max}]\}, \tag{120}$$

$$\sigma_{\mathrm{ess}}(\mathbf{L}) = i\mathbb{R} + m[\lambda_{\min}, \lambda_{\max}] \tag{121}$$

over the space $H^m_{\mathcal{F}}(\mathbb{T}^n)$.

In particular, the full spectral mapping theorem holds. Moreover, if $n = 2$, then from $\det \partial\varphi_t = 1$ we have $\lambda_{\min} = -\lambda_{\max}$. So, the essential spectrum of the 2D Euler and SQG equations is a solid band (annulus) symmetric with respect to the imaginary axis. This result was obtained previously by Latushkin, Friedlander and the author in [41, 40, 13] via an explicit construction of approximate eigenfunctions for each point in the band.

In the case $m = 0$ the identities (120), (121) become inclusions $\subseteq$ due to Theorem 5.3. These again turn into identities provided u has arbitrarily long trajectories [40].

7. Dynamical spectrum of the $b\xi^m$-cocycle

In this section we present the proofs of Theorems 6.1 and 6.2.

We introduce a scalar cocycle $\mathbf{X}^m$, the ξ^m-component of the $b\xi^m$-cocycle, by the rule

$$\mathbf{X}^m_t(x, \xi) = \left|\partial\varphi_t^{-\top}(x)\overline{\xi}\right|^m. \tag{122}$$

Notice that $\mathbf{X}^m$ is one-dimensional and is defined on the trivial scalar bundle over Ω^n (or $\mathbb{K}^n$). Hence, by Sacker and Sell's theorem its spectrum consists of a single interval given by

$$\Sigma_{\mathbf{X}^m} = m[\lambda_{\min}, \lambda_{\max}]. \tag{123}$$

We notice that the $b\xi^m$-cocycle is isomorphic to the tensor product of $\mathbf{X}^m$ and $\mathbf{B}$. Thus, from the results of [38] we obtain the following proposition.

Proposition 7.1. *The following inclusion holds:*

$$\Sigma_m \subset \Sigma + m[\lambda_{\min}, \lambda_{\max}]. \tag{124}$$

Certain estimates on the end-points of Σ_m follow trivially by definition or from (124). Let us denote

$$\begin{array}{ll} A_m = \mu_{\min} + m\lambda_{\min} & C_m = \mu_{\min} + m\lambda_{\max} \\ B_m = \mu_{\max} + m\lambda_{\min} & D_m = \mu_{\max} + m\lambda_{\max} \end{array} ;$$

for positive m, and

$$\begin{array}{ll} A_m = \mu_{\min} + m\lambda_{\max} & C_m = \mu_{\min} + m\lambda_{\min} \\ B_m = \mu_{\max} + m\lambda_{\max} & D_m = \mu_{\max} + m\lambda_{\min} \end{array} ;$$

for negative m.

Lemma 7.2. *The following estimates hold for all $m \in \mathbb{R}$:*

$$A_m \le \mu^m_{\min} \le \min\{B_m, C_m\}, \tag{125}$$

$$\max\{B_m, C_m\} \le \mu^m_{\max} \le D_m. \tag{126}$$

It is clear from these estimates that exponential stretching causes expansion of the spectrum Σ_m, as $m \to \infty$. On the other hand, if $\lambda_{\max} = 0$, it follows from (124) that $\Sigma_m \subset \Sigma$. Likewise, since $\mathbf{B} = \mathbf{X}^{-m}\mathbf{B}\mathbf{X}^m$, we have $\Sigma \subset \Sigma_m$. So, we have proved the following lemma.

Lemma 7.3. *The identity $\Sigma = \Sigma_m$ holds for all $m \in \mathbb{R}$ if and only if the flow φ has no exponential stretching of trajectories.*

Generally, Σ_m may have gaps. We can estimate the location of a possible gap in Σ_m using the Lyapunov exponents of the flow φ.

Proposition 7.4. *If m is positive, then*

$$[\mu^m_{\min}, \mu^m_{\max}] \backslash \Sigma_m \subset [\mu_{\min} + m\lambda_{\max} \, , \, \mu_{\max} + m\lambda_{\min}]. \tag{127}$$

If m is negative, then

$$[\mu^m_{\min}, \mu^m_{\max}] \backslash \Sigma_m \subset [\mu_{\min} + m\lambda_{\min} \, , \, \mu_{\max} + m\lambda_{\max}]. \tag{128}$$

Proof. Let $\mu \in [\mu^m_{\min}, \mu^m_{\max}]$ belong to the resolvent set of the $b\xi^m$-cocycle. Then there is an exponential dichotomy of the rescaled cocycle over $\mathbb{K}^n$. Let $\mathbf{P}$, ε and M be as in the definition (see Section 5.1). Since the projector $\mathbf{P}$ is non-trivial for all $(x, \xi) \in \mathbb{K}^n$, there exist b_1, $b_2 \in F(\xi)$ of unit norm such that

$$b_1 \in \operatorname{Rg} \mathbf{P}(x, \xi), \quad b_2 \in \operatorname{Ker} \mathbf{P}(x, \xi). \tag{129}$$

Thus, we have

$$|\mathbf{B}_t(x,\xi)b_1| \cdot |\partial\varphi_t^{-\top}(x)\xi|^m \le Me^{t(\mu-\varepsilon)}, \tag{130a}$$

$$|\mathbf{B}_t(x,\xi)b_2| \cdot |\partial\varphi_t^{-\top}(x)\xi|^m \ge M^{-1}e^{t(\mu+\varepsilon)}, \tag{130b}$$

for all $t \ge 0$.

Let λ be any end-point of $m[\lambda_{\min}, \lambda_{\max}]$. It is an exact Lyapunov exponent of $\mathbf{X}^m$ by [18]. So, there is a point $(x,\xi) \in \mathbb{K}^n$ such that

$$e^{(\lambda-\varepsilon)t} \le |\partial\varphi_t^{-\top}(x)\xi|^m \le e^{(\lambda+\varepsilon)t}, \tag{131}$$

for sufficiently large t. Combining (131) with the inequalities in (130), we obtain

$$|\mathbf{B}_t(x,\xi)b_1| \le Me^{t(\mu-\lambda)}, \tag{132a}$$

$$|\mathbf{B}_t(x,\xi)b_2| \ge M^{-1}e^{t(\mu-\lambda)}. \tag{132b}$$

Considering that the exponential type of $|\mathbf{B}_t(x,\xi)b_1|$ is not less than $\mu_{\min}$, while the exponential type of $|\mathbf{B}_t(x,\xi)b_2|$ does not exceed $\mu_{\max}$, (132) imply $\lambda + \mu_{\min} \le \mu \le \lambda + \mu_{\max}$. This proves the proposition. □

□

We can see from (127) and (128) that for large values of $|m|$ exponential stretching closes gaps in Σ_m. Precisely, $|m|$ has to be such that the end-points of the intervals on the left-hand sides of (127) and (128) meet. Thus, we obtain the following corollary, which in turn proves part 1) of Theorem 6.1.

Corollary 7.5. *If $\lambda_{\max} > 0$ and $|m| \ge \frac{\mu_{\max}-\mu_{\min}}{\lambda_{\max}-\lambda_{\min}}$, then Σ_m is connected.*

7.1. **The marginal spectrum.** Let us recall the following constants introduced in Section 6:

$$s = \sup_{k\in\mathbb{R}}\{\mu_{\min}^k\}, \quad S = \inf_{k\in\mathbb{R}}\{\mu_{\max}^k\}. \tag{133}$$

We define **marginal spectrum** of the $b\xi^m$-cocycle as the set

$$\mathsf{M}_m = \text{closure of } \Sigma_m \cap \left[(-\infty, s) \cup (S, +\infty)\right]. \tag{134}$$

We show that to any point of M_m there corresponds a Mañe point in the sense of Theorem 5.2, that is surrounded by non-periodic exponentially stretched orbits. But first, we find conditions which guarantee that M_m is non-empty, i.e. $\mu_{\max}^m > S$ and $\mu_{\min}^m < s$.

According to Lemma 7.3, a non-empty marginal spectrum is possible only if the flow φ has exponential stretching. In view of estimates (125) and (126), it suffices to have $B_m < \mu_{\min}$ and $C_m > \mu_{\max}$. We see that both inequalities are satisfied under the assumption Theorem 6.1, which

in view of Corollary 7.5 also implies connectedness of the two sides of M_m. Thus, we have obtained the following lemma.

Lemma 7.6. *Assume that $\lambda_{\max} > 0$ and $|m| > \frac{\mu_{\max}-\mu_{\min}}{\lambda_{\max}-\lambda_{\min}}$. Then M_m is non-empty and described by the following identity:*

$$\mathsf{M}_m = [\mu_{\min}^m, s] \cup [S, \mu_{\max}^m]. \tag{135}$$

We show now that to every point of the marginal spectrum there corresponds a Mañe point from Ω_0^n that satisfies the aperiodicity condition (ii) of Theorem 5.4.

Proposition 7.7. *For all $m \in \mathbb{R}$ we have*

$$\mathsf{M}_m = \mathsf{M}_m \cap \Sigma_m^{\perp}. \tag{136}$$

Furthermore, for any element of M_m there exists a Mañe point $(x_0, \xi_0) \in \Omega_0^n$, corresponding either to the rescaled $b\xi^m$-cocycle or to its dual, such that every neighborhood of x_0 intersects a non-periodic orbit.

Proof. We present the proof for the right margin only, since the assertion for the left margin follows by passing to the inverse cocycle.

We write $f(t) \lesssim g(t)$ to signify that the exponential type of $f(t)$ is less than the exponential type of $g(t)$.

If $\mu_{\max}^m \leq S$, then the identity (136) is trivial. So, let $\mu_{\max}^m > \mu_{\max}^k$ for some $k \in \mathbb{R}$, and we fix any $\lambda \in \Sigma_m$ with $\lambda > \mu_{\max}^k$.

We proceed in several steps considering all possible cases.

Suppose $m > k$. According to Theorem 5.2 there is a Mañe sequence either for the $b\xi^m$-cocycle or for its dual.

Let us assume the former. Then by (90a) there is a sequence $\{(x_n, \xi_n), b_n\}_{n=1}^{\infty}$, with $b_n \in F(\xi_n)$, such that

$$|b(n)||\xi(n)|^m \gtrsim e^{n\lambda}.$$

Here we denote

$$b(n) = \mathbf{B}_n(x_n, \xi_n) b_n,$$
$$\xi(n) = \partial\varphi_n^{-\top}(x_n)\xi_n.$$

At the same time, by the definition of $\mu_{\max}^k$, one has

$$|b(n)||\xi(n)|^m = |b(n)||\xi(n)|^k |\xi(n)|^{m-k} \lesssim e^{n\mu_{\max}^k} |\xi(n)|^{m-k}.$$

This shows that $\xi(n)$ is growing exponentially as $n \to \infty$.

Let (x_0, ξ_0), $|\xi_0| = 1$, be the corresponding Mañe point constructed from the sequence $\{(x_n, \overline{\xi}_n)\}_{n=1}^{\infty} \subset \mathbb{K}^n$ as in Section 5.1. Thus, by

construction,

$$x_0 = \lim_{n\to\infty} \varphi_n(x_n),$$

$$\xi_0 = \lim_{n\to\infty} \frac{\xi(n)}{|\xi(n)|}.$$

It follows that

$$u(x_0)\cdot\xi_0 = \lim_{n\to\infty} u(\varphi_n(x_n))\cdot\frac{\xi(n)}{|\xi(n)|} = \lim_{n\to\infty} u(x_n)\cdot\frac{\xi_n}{|\xi(n)|} = 0.$$

Thus, (x_0,ξ_0) belongs to Ω_0^n, and hence, $\lambda \in \mathsf{M}_m^\perp$.

Suppose now there is a Mañe sequence for the dual cocycle, while still $m > k$. Then there exists a Mañe point $(x_0,\xi_0) \in \mathbb{K}^n$ and Mañe vector $b_0 \in F(\xi_0)$. It is straightforward to check the the dual cocycle is given by

$$\left|\partial\varphi_{-t}^{-\top}(x)\overline{\xi}\right|^{-m} \mathbf{B}_{-t}^{-*}(x,\xi). \tag{137}$$

Thus, we have

$$|\mathbf{B}_{-t}^{-*}(x_0,\xi_0)b_0|\cdot|\partial\varphi_{-t}^{-\top}(x_0)\xi_0|^{-m} \le Ce^{\lambda t}, \quad t\in\mathbb{R}.$$

We can see that the same point is Mañe for the inverse dual cocycle rescaled by $-\lambda$, so that

$$|\mathbf{B}_{t}^{-*}(x_0,\xi_0)b_0|\cdot|\partial\varphi_{t}^{-\top}(x_0)\xi_0|^{-m} \le Ce^{-\lambda t}, \quad t\in\mathbb{R}.$$

Let us reconstruct the Mañe sequence from (x_0,ξ_0,b_0) as in Section 5.1. We obtain a sequence $\{(x_n,\xi_n),b_n\}_{n=1}^\infty$, with $b_n\in F(\xi_n)$, such that, in particular,

$$|b^*(n)||\xi(n)|^{-m} \lesssim e^{-\lambda n}, \tag{138}$$

where $\xi(n)$ as before, and

$$b^*(n) = \mathbf{B}_n^{-*}(x_n,\xi_n)b_n.$$

Recall that the inverse dual cocycle has the opposite spectrum. Thus, in addition to (138), we obtain

$$|b^*(n)||\xi(n)|^{-m} \gtrsim e^{-n\mu_{\max}^k}|\xi(n)|^{k-m}. \tag{139}$$

Combining (138) and (139) we see again that $\xi(n)$ is exponentially increasing.

We find a point $(x_0',\xi_0') \in \Omega_0^n$ constructed from this sequence, which is a Mañe point for the inverse dual cocycle rescaled by $-\lambda$. As before, this same point is Mañe for the dual cocycle rescaled by λ. So, λ belongs to the dynamical spectrum of the dual cocycle, and hence, to the spectrum of the original $b\xi^m$-cocycle on Ω_0^n. Thus, $\lambda\in\Sigma_m^\perp$.

Let us now consider the case when $m < k$. If a Mañe sequence exists for the $b\xi^m$-cocycle itself, then as in the previous paragraph we can find a Mañe sequence for the inverse cocycle corresponding to $-\lambda$. This implies

$$e^{-n\mu^k_{\max}}|\xi(-n)|^{m-k} \lesssim |b(-n)||\xi(-n)|^m \lesssim e^{-n\lambda}.$$

So, $|\xi(-n)|$ is exponentially increasing and we finish the proof as before.

Finally, if the dual to the $b\xi^m$-cocycle has a Mañe sequence, then we obtain

$$e^{n\mu^k_{\max}}|\xi(-n)|^{k-m} \gtrsim |b^*(-n)||\xi(-n)|^{-m} \gtrsim e^{n\lambda}.$$

Again, $|\xi(-n)|$ is increasing exponentially.

We have considered all possible cases. This finishes the proof of (136).

Let us notice that in each of the above cases we have obtained a Mañe point $(x_0, \xi_0) \in \Omega^n_0$ such that $\xi(t, x_0, \xi_0)$ has a non-trivial exponential type either in the forward or in the backward direction. If the φ-orbit through x_0 is not periodic, then the "furthermore" statement is trivial. Otherwise, we use the Stable Manifold Theorem [33] to find stable and unstable manifolds in a neighborhood of the orbit. Either of the manifolds consists of non-periodic orbits. □ □

Corollary 7.8. *If* $\lambda_{\max} > 0$ *and* $|m| > \frac{\mu_{\max}-\mu_{\min}}{\lambda_{\max}-\lambda_{\min}}$, *then*

$$\Sigma^\perp_m = \Sigma_m = [\mu^m_{\min}, \mu^m_{\max}] \tag{140}$$

Proof. The end-points of Σ_m and $\Sigma^\perp_m$ coincide due to Proposition 7.7. On the other hand, the analogue of Proposition 7.4 on Ω^n_0 is straightforward. It only suffices to notice that if $\lambda_{\max} > 0$, then the point (x, ξ) in formula (131) must belong to Ω^n_0 by conservation of the Hamiltonian $u(x) \cdot \xi$.

As in Corollary 7.5 this shows that $\Sigma^\perp_m$ is connected, and hence, (140) holds. □ □

With the help of Theorem 5.4 and Theorem 5.6 the results of this section prove Theorems 6.1 and 6.2 completely.

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